Strongly anisotropic diffusion problems; asymptotic analysis

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Abstract

The subject matter of this paper concerns anisotropic diffusion equations: we consider heat equations whose diffusion matrices have disparate eigenvalues. We determine first and second order approximations, we study the well-posedness of them and establish convergence results. The analysis relies on averaging techniques, which have been used previously for studying transport equations whose advection fields have disparate components.

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1 Introduction

Many real life applications lead to highly anisotropic diffusion equations: flows in porous media, quasi-neutral plasmas, microscopic transport in magnetized plasmas [7], plasma thrusters, image processing [18], [23], thermal properties of crystals [13], [19]. In this paper we investigate the behavior of the solutions for heat equations whose diffusion becomes very high along some direction. We consider the problem

$$\partial_t u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) - \frac{1}{\varepsilon}\mathrm{div}_y(b(y)\otimes b(y)\nabla_y u^\varepsilon) = 0,\quad (t,y)\in\mathbb{R}_+\times\mathbb{R}^m \tag{1}$$

$$u^\varepsilon(0,y) = u^\varepsilon_{\mathrm{in}}(y),\quad y\in\mathbb{R}^m \tag{2}$$

where $D(y)\in\mathcal{M}_m(\mathbb{R})$ and $b(y)\in\mathbb{R}^m$ are smooth given matrix field and vector field on $\mathbb{R}^m$, respectively. For any two vectors ξ,η, the notation $\xi\otimes\eta$ stands for the matrix whose entry

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(i,j) is $\xi_i \eta_j$, and for any two matrices A, B the notation $A : B$ stands for trace$({}^t AB) = A_{ij}B_{ij}$ (using Einstein summation convention). We assume that at any $y \in \mathbb{R}^m$ the matrix $D(y)$ is symmetric and $D(y) + b(y) \otimes b(y)$ is positive definite

$$ {}^t D(y) = D(y), \;\; \exists\, d > 0 \;\text{ such that }\; D(y)\xi \cdot \xi + (b(y)\cdot \xi)^2 \geq d\,|\xi|^2, \;\; \xi \in \mathbb{R}^m, \;\; y \in \mathbb{R}^m. \quad (3) $$

The vector field $b(y)$, to which the anisotropy is aligned, is supposed divergence free *i.e.,* $\mathrm{div}_y b = 0$. We intend to analyse the behavior of (1), (2) for small ε, let us say $0 < \varepsilon \leq 1$, in which cases $D(y) + \frac{1}{\varepsilon} b(y) \otimes b(y)$ remains positive definite

$$ D(y)\xi\cdot\xi + \frac{1}{\varepsilon}(b(y)\cdot\xi)^2 \geq D(y)\xi\cdot\xi + (b(y)\cdot\xi)^2 \geq d|\xi|^2, \;\; \xi \in \mathbb{R}^m, \;\; y \in \mathbb{R}^m. \quad (4) $$

If $(u^\varepsilon_{\mathrm{in}})_\varepsilon$ remain in a bounded set of $L^2(\mathbb{R}^m)$, then $(u^\varepsilon)_\varepsilon$ remain in a bounded set of $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ since, for any $t \in \mathbb{R}_+$ we have, thanks to (4)

$$ \begin{aligned} \frac{1}{2}\int_{\mathbb{R}^m} (u^\varepsilon(t,y))^2 \,\mathrm{d}y + d\int_0^t\!\!\int_{\mathbb{R}^m} |\nabla_y u^\varepsilon(s,y)|^2 \,\mathrm{d}y\mathrm{d}s &\leq \frac{1}{2}\int_{\mathbb{R}^m} (u^\varepsilon(t,y))^2 \,\mathrm{d}y \\ &+ \int_0^t\!\!\int_{\mathbb{R}^m} \left\{ D(y) + \frac{1}{\varepsilon} b(y)\otimes b(y)\right\} : \nabla_y u^\varepsilon(s,y) \otimes \nabla_y u^\varepsilon(s,y)\,\mathrm{d}y\mathrm{d}s \\ &= \frac{1}{2}\int_{\mathbb{R}^m} (u^\varepsilon_{\mathrm{in}}(y))^2 \,\mathrm{d}y. \end{aligned} $$

In particular, when $\varepsilon \searrow 0$, $(u^\varepsilon)_\varepsilon$ converges, at least weakly $\star$ in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ towards some limit $u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$. Notice that the explicit methods are not well adapted for the numerical approximation of (1), (2) when $\varepsilon \searrow 0$, since the CFL condition leads to severe time step constraints like

$$ \frac{d}{\varepsilon}\frac{\Delta t}{|\Delta y|^2} \leq \frac{1}{2} $$

where Δt is the time step and Δy is the grid spacing. In such cases implicit methods are desirable [2], [21]. For the numerical resolution of diffusion equations on distorted grids we refer to [17], [16], [20]. Finite volume methods have been discussed in [14], [1]. Recent results concerning anisotropic elliptic problems and non linear heat equations were obtained in [11], [12], [15].

In plasma physics, the collision operator gives rise to anisotropic diffusion in velocity space due to the interaction between particles and waves [22]. The applications we have in mind concern the magnetic confinement. This analysis is required when studying the energy (temperature) anisotropic diffusion inside a tokamak [9]. In that case the diffusion along the magnetic lines dominates the diffusion along the other directions and the temperature satisfies a heat equation like (1) where $m \in \{2,3\}$, $b(y)$ stands for the magnetic field and

$D(y)$ is the diffusion matrix along the perpendicular directions to $b(y)$

$$D(y) = d\left(I_m - \frac{b(y)\otimes b(y)}{|b(y)|^2}\right).$$

Rather than solving (1), (2) for small $\varepsilon > 0$, we concentrate on the limit model satisfied by the limit solution $u = \lim_{\varepsilon \searrow 0} u^\varepsilon$. We will see that the limit model is still a parabolic problem, decreasing the $L^2(\mathbb{R}^m)$ norm and satisfying the maximum principle. At least formally, the limit solution u is the dominant term of the expansion

$$u^\varepsilon = u + \varepsilon u^1 + \varepsilon^2 u^2 + ... \tag{5}$$

Plugging the Ansatz (5) into (1) leads to

$$\operatorname{div}_y(b\otimes b\nabla_y u) = 0, \quad (t,y)\in\mathbb{R}_+\times\mathbb{R}^m \tag{6}$$

$$\partial_t u - \operatorname{div}_y(D\nabla_y u) - \operatorname{div}_y(b\otimes b\nabla_y u^1) = 0, \quad (t,y)\in\mathbb{R}_+\times\mathbb{R}^m \tag{7}$$

$$\vdots$$

Under suitable hypotheses on b, the constraint (6) says that at any time $t\in\mathbb{R}_+$, $b\cdot\nabla_y u = 0$ (see Proposition 3.3), or equivalently $u(t,\cdot)$ remains constant along the flow of b, see (16)

$$u(t, Y(s;y)) = u(t,y), \quad s\in\mathbb{R}, \quad y\in\mathbb{R}^m.$$

The closure for u comes by eliminating u^1 in (7), combined with the fact that (6) holds true at any time $t\in\mathbb{R}_+$. The symmetry of the operator $\operatorname{div}_y(b\otimes b\nabla_y)$ implies that $\partial_t u - \operatorname{div}_y(D\nabla_y u)$ belongs to $(\ker(b\cdot\nabla_y))^\perp$ and therefore we obtain the weak formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} u(t,y)\varphi(y)\,\mathrm{d}y + \int_{\mathbb{R}^m} D\nabla_y u(t,y)\cdot\nabla_y\varphi(y)\,\mathrm{d}y = 0, \quad \varphi\in H^1(\mathbb{R}^m)\cap\ker(b\cdot\nabla_y). \tag{8}$$

The above formulation is not satisfactory, since the choice of test functions is constrained by (6); (8) is useless for numerical simulation. A more convenient situation is to reduce (8) to another problem, by removing the constraint (6). The method we employ here is related to the averaging technique which has been used to handle transport equations with disparate advection fields [3], [4], [5], [6]

$$\partial_t u^\varepsilon + a(t,y)\cdot\nabla_y u^\varepsilon + \frac{1}{\varepsilon} b(y)\cdot\nabla_y u^\varepsilon = 0, \quad (t,y)\in\mathbb{R}_+\times\mathbb{R}^m \tag{9}$$

$$u^\varepsilon(0,y) = u^\varepsilon_{\mathrm{in}}(y), \quad y\in\mathbb{R}^m. \tag{10}$$

Using the same Ansatz (5) we obtain as before that $b\cdot\nabla_y u(t,\cdot) = 0, t\in\mathbb{R}_+$ and the closure for u writes

$$\operatorname{Proj}_{\ker(b\cdot\nabla_y)}\{\partial_t u + a\cdot\nabla_y u\} = 0 \tag{11}$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} u(t,y)\varphi(y)\,\mathrm{d}y - \int_{\mathbb{R}^m} u(t,y)\; a\cdot\nabla_y\varphi\,\mathrm{d}y = 0 \tag{12}$$

for any smooth function satisfying the constraint $b\cdot\nabla_y\varphi = 0$. The method relies on averaging since the projection on $\ker(b\cdot\nabla_y)$ coincides with the average along the flow of b, cf. Proposition 3.1. As u satisfies the constraint $b\cdot\nabla_y u = 0$, it is easily seen that $\mathrm{Proj}_{\ker(b\cdot\nabla_y)}\partial_t u = \partial_t u$. A simple case to start with is when the transport operator $a\cdot\nabla_y$ and $b\cdot\nabla_y$ commute *i.e.,* $[b\cdot\nabla_y, a\cdot\nabla_y] = 0$. In this case $a\cdot\nabla_y$ leaves invariant the subspace of the constraints, implying that $\mathrm{Proj}_{\ker(b\cdot\nabla_y)}\{a\cdot\nabla_y u\} = a\cdot\nabla_y u$. Therefore (11) reduces to a transport equation and it is easily seen that this equation propagates the constraint, which allows us to remove it. Things happen similarly when the transport operators $a\cdot\nabla_y, b\cdot\nabla_y$ do not commute, but the transport operator of the limit model may change. In [4] we prove that there is a transport operator $A\cdot\nabla_y$, commuting with $b\cdot\nabla_y$, such that for any $u\in\ker(b\cdot\nabla_y)$ we have

$$\mathrm{Proj}_{\ker(b\cdot\nabla_y)}\{a\cdot\nabla_y u\} = A\cdot\nabla_y u.$$

Once we have determined the field A, (11) can be replaced by $\partial_t u + A\cdot\nabla_y u = 0$, which propagates the constraint $b\cdot\nabla_y u(t) = 0$ as well.

Coming back to the formulation (8), we are looking for a matrix field $\tilde{D}(y)$ such that $\mathrm{div}_y(\tilde{D}\nabla_y)$ commutes with $b\cdot\nabla_y$ and

$$\mathrm{Proj}_{\ker(b\cdot\nabla_y)}\{\mathrm{div}_y(D(y)\nabla_y u)\} = \mathrm{div}_y(\tilde{D}(y)\nabla_y u),\;\; u\in\ker(b\cdot\nabla_y).$$

We will see that, under suitable hypotheses, it is possible to find such a matrix field $\tilde{D}$, and therefore (8) reduces to the parabolic model

$$\partial_t u - \mathrm{div}_y(\tilde{D}(y)\nabla_y u) = 0,\;\; (t,y)\in\mathbb{R}_+\times\mathbb{R}^m. \tag{13}$$

The matrix field $\tilde{D}$ will appear as the orthogonal projection of the matrix field D (with respect to some scalar product to be determined) on the subspace of matrix fields A satisfying $[b\cdot\nabla_y, \mathrm{div}_y(A\nabla_y)] = 0$. The field $\tilde{D}$ inherits the properties of D, like symmetry, positivity, etc.

Our paper is organized as follows. The main results are presented in Section 2. Section 3 is devoted to the interplay between the average operator and first and second order linear differential operators. In particular we justify the existence of the *averaged* matrix field $\tilde{D}$ associated to any field D of symmetric, positive matrices. The first order approximation is justified in Section 4 and the second order approximation is discussed in Section 5. Some examples, in particular the application to the magnetic confinement, are treated in Section 6. Several technical proofs are gathered in Appendix A.

2 Presentation of the models and main results

We assume that the vector field $b : \mathbb{R}^m \to \mathbb{R}^m$ is smooth and divergence free

$$b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m), \;\; \mathrm{div}_y b = 0 \tag{14}$$

with linear growth

$$\exists\, C > 0 \;\text{ such that }\; |b(y)| \leq C(1 + |y|), \;\; y \in \mathbb{R}^m. \tag{15}$$

We denote by $Y(s; y)$ the characteristic flow associated to b

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = b(Y(s; y)), \;\; Y(0; y) = y, \;\; s \in \mathbb{R}, \;\; y \in \mathbb{R}^m. \tag{16}$$

Under the above hypotheses, this flow has the regularity $Y \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^m)$ and is measure preserving.

We concentrate on matrix fields $A(y) \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ such that $[b(y) \cdot \nabla_y, \mathrm{div}_y(A(y)\nabla_y)] = 0$, let us say in $\mathcal{D}'(\mathbb{R}^m)$. We check that the commutator between $b \cdot \nabla_y$ and $\mathrm{div}_y(A\nabla_y)$ writes cf. Proposition 3.7

$$[b(y) \cdot \nabla_y, \mathrm{div}_y(A(y)\nabla_y)] = \mathrm{div}_y([b, A]\nabla_y) \;\text{ in }\; \mathcal{D}'(\mathbb{R}^m)$$

where the bracket between b and A is given by the matrix field

$$[b, A] := (b \cdot \nabla_y)A - \partial_y b A(y) - A(y)\, {}^t\partial_y b, \;\; y \in \mathbb{R}^m$$

with $((b \cdot \nabla_y)A)_{ij} = (b \cdot \nabla_y)A_{ij}, \; i, j \in \{1, ..., m\}$. Several characterizations for the solutions of $[b, A] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ are indicated in the Propositions 3.8, 3.9, among which

$$A(Y(s; y)) = \partial_y Y(s; y) A(y)\, {}^t\partial_y Y(s; y), \;\; s \in \mathbb{R}, \;\; y \in \mathbb{R}^m. \tag{17}$$

We assume that there is a matrix field $P(y)$ such that

$${}^tP = P, \;\; P(y)\xi \cdot \xi > 0, \;\; \xi \in \mathbb{R}^m, \;\; y \in \mathbb{R}^m, \;\; P^{-1}, P \in L^2_{\mathrm{loc}}(\mathbb{R}^m), \;\; [b, P] = 0 \text{ in } \mathcal{D}'(\mathbb{R}^m). \tag{18}$$

Observe that any vector field c in involution with b *i.e.*, $(b \cdot \nabla_y)c - \partial_y b\; c = 0$ let us say in $\mathcal{D}'(\mathbb{R}^m)$, provides a symmetric matrix field $P_c(y) = c(y) \otimes c(y)$ satisfying $[b, P_c] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. Indeed, thanks to Proposition 3.4 we have

$$c(Y(s; y)) = \partial_y Y(s; y) c(y), \;\; s \in \mathbb{R}, y \in \mathbb{R}^m$$

and therefore (17) holds true

$$\begin{aligned}
P_c(Y(s;y)) &= c(Y(s;y)) \otimes c(Y(s;y)) \\
&= (\partial_y Y(s;y)c(y)) \otimes (\partial_y Y(s;y)c(y)) \\
&= \partial_y Y(s;y)(c(y) \otimes c(y))\, {}^t\partial_y Y(s;y) \\
&= \partial_y Y(s;y) P_c(y)\, {}^t\partial_y Y(s;y), \quad s \in \mathbb{R}, y \in \mathbb{R}^m.
\end{aligned}$$

When a family $\{c_i\}_{1\leq i\leq m}$ of vector fields in involution with b is available, and $\{c_i(y)\}_{1\leq i\leq m}$ form a basis of $\mathbb{R}^m$ at any point $y \in \mathbb{R}^m$, it is easily seen that the symmetric matrix field $P(y) = \sum_{i=1}^m c_i(y) \otimes c_i(y)$ is positive definite and satisfies $[b, P] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$.

We introduce the set

$$H_Q = \{A = A(y) \ : \ \int_{\mathbb{R}^m} Q(y)A(y) : A(y)Q(y)\ \mathrm{d}y < +\infty\}$$

where $Q = P^{-1}$, and the scalar product (see Section 3.3)

$$(A, B)_Q = \int_{\mathbb{R}^m} QA : BQ\ \mathrm{d}y, \quad A, B \in H_Q.$$

The equality (17) suggests to introduce the family of applications $G(s) : H_Q \to H_Q$, $s \in \mathbb{R}$, $G(s)A = (\partial_y Y)^{-1}(s;\cdot)A(Y(s;\cdot))\ {}^t(\partial_y Y)^{-1}(s;\cdot)$ which is a C^0-group of unitary operators on H_Q cf. Proposition 3.12. This allows us to introduce L, the infinitesimal generator of $(G(s))_{s\in\mathbb{R}}$. The key points are that L becomes skew-adjoint on the weighted L^2 space H_Q and that its kernel coincides with $\{A \in H_Q \subset L^1_{\mathrm{loc}}(\mathbb{R}^m) : [b, A] = 0 \text{ in } \mathcal{D}'(\mathbb{R}^m)\}$ cf. Proposition 3.13. The averaged matrix field denoted $\langle D\rangle_Q$, associated to any $D \in H_Q$ appears as the long time limit of the solution of

$$\partial_t A - L(L(A)) = 0, \quad t \in \mathbb{R}_+ \tag{19}$$

$$A(0) = D. \tag{20}$$

The condition $D \in H_Q$ comes since we intend to use the C^0-group theory in L^2 spaces (weighted by Q). In particular H_Q contains any matrix field D bounded, compactly supported in $\mathbb{R}^m$. The notation $\langle\cdot\rangle$ stands for the orthogonal projection (in $L^2(\mathbb{R}^m)$) on $\ker(b \cdot \nabla_y)$.

Theorem 2.1 *Assume that* (14), (15), (18) *hold true. Then for any $D \in H_Q \cap L^\infty(\mathbb{R}^m)$ the solution of* (19), (20) *converges weakly in H_Q as $t \to +\infty$ towards the orthogonal projection of D on* $\ker L$

$$\lim_{t\to+\infty} A(t) = \langle D\rangle_Q \quad \textit{weakly in} \ \ H_Q, \ \ \langle D\rangle_Q := \mathrm{Proj}_{\ker L} D.$$

If D is symmetric and positive, then so is the limit $\langle D\rangle_Q = \lim_{t\to+\infty} A(t)$, and satisfies

$$L(\langle D\rangle_Q) = 0, \;\; \nabla_y u \cdot \langle D\rangle_Q \nabla_y v = \langle \nabla_y u \cdot D \nabla_y v\rangle\,, \;\; u, v \in H^1(\mathbb{R}^m) \cap \ker(b\cdot\nabla_y) \tag{21}$$

$$\left\langle \nabla_y u \cdot \langle D\rangle_Q \nabla_y (b\cdot\nabla_y \psi)\right\rangle = 0, \;\; u \in H^1(\mathbb{R}^m) \cap \ker(b\cdot\nabla_y), \;\; \psi \in C^2_c(\mathbb{R}^m). \tag{22}$$

The first order approximation (for initial data not necessarily well prepared) is justified by

Theorem 2.2 *Assume that* (14), (15), (18), (49) *hold true and that D is a field of symmetric positive matrices, which belongs to H_Q. Consider a family of initial conditions $(u^\varepsilon_{\text{in}})_\varepsilon \subset L^2(\mathbb{R}^m)$ such that $(\langle u^\varepsilon_{\text{in}}\rangle)_\varepsilon$ converges weakly in $L^2(\mathbb{R}^m)$, as $\varepsilon \searrow 0$, towards some function u_{in}. We denote by u^ε the solution of* (1), (2) *and by u the solution of*

$$\partial_t u - \operatorname{div}_y(\langle D\rangle_Q \nabla_y u) = 0, \;\; t\in\mathbb{R}_+, \;\; y \in \mathbb{R}^m \tag{23}$$

$$u(0,y) = u_{\text{in}}(y), \;\; y\in\mathbb{R}^m \tag{24}$$

where $\langle D\rangle_Q$ is associated to D, cf. Theorem 2.1. Then we have the convergences

$$\lim_{\varepsilon\searrow 0} u^\varepsilon = u \;\; \textit{weakly}\star \;\; \textit{in } L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$$

$$\lim_{\varepsilon\searrow 0} \nabla_y u^\varepsilon = \nabla_y u \;\; \textit{weakly in } L^2(\mathbb{R}_+; L^2(\mathbb{R}^m)).$$

The derivation of the second order approximation is more complicated and requires the computation of some other matrix fields. For simplicity, we content ourselves to formal results. The crucial point is to introduce the decomposition given by

Theorem 2.3 *Assume that* (14), (15), (18), (49) *hold true and that L has closed range. Then, for any field of symmetric matrices $D \in H_Q$, there is a unique field of symmetric matrices $F \in \operatorname{dom}(L^2)\cap(\ker L)^\perp$ such that*

$$-\operatorname{div}_y(D\nabla_y) = -\operatorname{div}_y(\langle D\rangle_Q \nabla_y) + \operatorname{div}_y(L^2(F)\nabla_y)$$

that is

$$\begin{aligned}
&\int_{\mathbb{R}^m} D\nabla_y u\cdot\nabla_y v \,\mathrm{d}y - \int_{\mathbb{R}^m} \langle D\rangle_Q \nabla_y u \cdot \nabla_y v\,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} L(F)\nabla_y u\cdot\nabla_y(b\cdot\nabla_y v)\,\mathrm{d}y + \int_{\mathbb{R}^m} L(F)\nabla_y(b\cdot\nabla_y u)\cdot\nabla_y v\,\mathrm{d}y \\
&= -\int_{\mathbb{R}^m} F\nabla_y(b\cdot\nabla_y(b\cdot\nabla_y u))\cdot\nabla_y v\,\mathrm{d}y - 2\int_{\mathbb{R}^m} F\nabla_y(b\cdot\nabla_y u)\cdot\nabla_y(b\cdot\nabla_y v)\,\mathrm{d}y \\
&- \int_{\mathbb{R}^m} F\nabla_y u\cdot\nabla_y(b\cdot\nabla_y(b\cdot\nabla_y v))\,\mathrm{d}y
\end{aligned}$$

for any $u, v \in C^3_c(\mathbb{R}^m)$.

After some computations we obtain, at least formally, the following model, replacing the hypothesis (18) by the stronger one: there is a matrix field $R(y)$ such that

$$\det R(y) \neq 0,\ y \in \mathbb{R}^m,\ Q = {}^t RR \text{ and } P = Q^{-1} \in L^2_{\mathrm{loc}}(\mathbb{R}^m),\ (b\cdot\nabla_y)R + R\partial_y b = 0 \text{ in } \mathcal{D}'(\mathbb{R}^m). \tag{25}$$

The condition $(b \cdot \nabla_y)R + R\partial_y b = 0$ says that the columns of R^{-1} form a family of m independent vector fields in involution with respect to b (see Remark 5.2).

Theorem 2.4 *Assume that* (14), (15), (29), (49), (25) *hold true and that D is a field of symmetric positive matrices which belongs to $H_Q \cap L^\infty(\mathbb{R}^m)$. Consider a family of initial conditions $(u^\varepsilon_{\mathrm{in}})_\varepsilon \subset L^2(\mathbb{R}^m)$ such that $(\frac{\langle u^\varepsilon_{\mathrm{in}}\rangle - u_{\mathrm{in}}}{\varepsilon})_{\varepsilon>0}$ converges weakly in $L^2(\mathbb{R}^m)$, as $\varepsilon \searrow 0$, towards a function v_{in}, for some function $u_{\mathrm{in}} \in \ker(b\cdot\nabla_y)$. Then, a second order approximation for* (1) *is provided by*

$$\partial_t \tilde{u}^\varepsilon - \mathrm{div}_y(\langle D\rangle_Q \nabla_y \tilde{u}^\varepsilon) + \varepsilon[\mathrm{div}_y(\langle D\rangle_Q \nabla_y), \mathrm{div}_y(F\nabla_y)]\tilde{u}^\varepsilon - \varepsilon S(\tilde{u}^\varepsilon) = 0,\ \ (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m \tag{26}$$

$$\tilde{u}^\varepsilon(0,y) = u_{\mathrm{in}}(y) + \varepsilon(v_{\mathrm{in}}(y) + w_{\mathrm{in}}(y)),\ \ w_{\mathrm{in}} = \mathrm{div}_y(F\nabla_y u_{\mathrm{in}}),\ \ y \in \mathbb{R}^m \tag{27}$$

for some fourth order linear differential operator S, see Proposition 5.3, and the matrix field F given by Theorem 2.3.

3 The average operator

We assume that the vector field $b : \mathbb{R}^m \to \mathbb{R}^m$ satisfies (14), (15). We consider the linear operator $u \to b\cdot\nabla_y u = \mathrm{div}_y(ub)$ in $L^2(\mathbb{R}^m)$, whose domain is defined by

$$\mathrm{dom}(b\cdot\nabla_y) = \{u \in L^2(\mathbb{R}^m)\ :\ \mathrm{div}_y(ub) \in L^2(\mathbb{R}^m)\}.$$

It is well known that

$$\ker(b\cdot\nabla_y) = \{u \in L^2(\mathbb{R}^m)\ :\ u(Y(s;\cdot)) = u(\cdot),\ s \in \mathbb{R}\}.$$

The orthogonal projection on $\ker(b\cdot\nabla_y)$ (with respect to the scalar product of $L^2(\mathbb{R}^m)$), denoted by $\langle\cdot\rangle$, reduces to average along the characteristic flow Y cf. [4] Propositions 2.2, 2.3. The fact that the average along the characteristic flow belongs to $\ker(b\cdot\nabla_y)$ is easily seen. For any point $y \in \mathbb{R}^m$, the average along the characteristic issued from y depends only on the invariants of this characteristic and not on the particular point y of it. Therefore, the average depends only on the invariants of the flow, and thus it belongs to $\ker(b\cdot\nabla_y)$.

Proposition 3.1 *For any function $u \in L^2(\mathbb{R}^m)$ the family $\langle u \rangle_T := \frac{1}{T}\int_0^T u(Y(s;\cdot))\mathrm{d}s, T > 0$ converges strongly in $L^2(\mathbb{R}^m)$, when $T \to +\infty$, towards the orthogonal projection of u on $\ker(b\cdot\nabla_y)$*

$$\lim_{T\to+\infty} \langle u \rangle_T = \langle u \rangle\ ,\ \ \langle u \rangle \in \ker(b\cdot\nabla_y) \text{ and } \int_{\mathbb{R}^m} (u - \langle u \rangle)\varphi \,\mathrm{d}y = 0,\ \forall\ \varphi \in \ker(b\cdot\nabla_y).$$

Since $b\cdot\nabla_y$ is antisymmetric, one gets easily

$$\overline{\text{Range }(b\cdot\nabla_y)} = (\ker(b\cdot\nabla_y))^{\perp} = \ker(\text{Proj}_{\ker(b\cdot\nabla_y)}) = \ker\langle\cdot\rangle\,. \tag{28}$$

Remark 3.1 *If $u \in L^2(\mathbb{R}^m)$ satisfies $\int_{\mathbb{R}^m} u(y) b\cdot\nabla_y\psi\,\mathrm{d}y = 0, \forall\ \psi \in C^1_c(\mathbb{R}^m)$ and $\int_{\mathbb{R}^m} u\varphi\,\mathrm{d}y = 0, \forall\ \varphi \in \ker(b\cdot\nabla_y)$, then $u = 0$. Indeed, as $u \in L^2(\mathbb{R}^m) \subset L^1_{\text{loc}}(\mathbb{R}^m)$, the first condition says that $b\cdot\nabla_y u = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ and thus $u \in \ker(b\cdot\nabla_y)$. Using now the second condition with $\varphi = u$ one gets $\int_{\mathbb{R}^m} u^2\,\mathrm{d}y = 0$ and thus $u = 0$.*

In the particular case when Range $(b\cdot\nabla_y)$ is closed, which is equivalent to the Poincaré inequality (cf. [8] pp. 29)

$$\exists\ C_P > 0\ :\ \left(\int_{\mathbb{R}^m} (u - \langle u \rangle)^2\,\mathrm{d}y\right)^{1/2} \le C_P \left(\int_{\mathbb{R}^m} (b\cdot\nabla_y u)^2\,\mathrm{d}y\right)^{1/2},\ \ u \in \text{dom}(b\cdot\nabla_y) \tag{29}$$

(28) implies the solvability condition

$$\exists\ u \in \text{dom}(b\cdot\nabla_y)\ \text{ such that }\ b\cdot\nabla_y u = v\ \text{ iff }\ \langle v \rangle = 0.$$

If $\|\cdot\|$ stands for the $L^2(\mathbb{R}^m)$ norm we have

Proposition 3.2 *Under the hypothesis* (29)*, $b\cdot\nabla_y$ restricted to $\ker\langle\cdot\rangle$ is one to one map onto $\ker\langle\cdot\rangle$. Its inverse, denoted $(b\cdot\nabla_y)^{-1}$, belongs to $\mathcal{L}(\ker\langle\cdot\rangle, \ker\langle\cdot\rangle)$ and*

$$\|(b\cdot\nabla_y)^{-1}\|_{\mathcal{L}(\ker\langle\cdot\rangle,\ker\langle\cdot\rangle)} \le C_P.$$

Another operator which will play a crucial role is $\mathcal{T} = -\text{div}_y(b\otimes b\nabla_y)$ whose domain is

$$\text{dom}(\mathcal{T}) = \{u \in \text{dom}(b\cdot\nabla_y)\ :\ b\cdot\nabla_y u \in \text{dom}(b\cdot\nabla_y)\}.$$

The operator $\mathcal{T}$ is self-adjoint and under the previous hypotheses, has the same kernel and range as $b\cdot\nabla_y$.

Proposition 3.3 *Under the hypotheses* (14)*,* (15)*,* (29) *the operator $\mathcal{T}$ satisfies*

$$\ker\mathcal{T} = \ker(b\cdot\nabla_y),\ \ \text{Range }\mathcal{T} = \text{Range }(b\cdot\nabla_y) = \ker\langle\cdot\rangle$$

and $\|u - \langle u \rangle\| \le C_P^2\|\mathcal{T}u\|, u \in \text{dom}(\mathcal{T})$.

Proof. Obviously $\ker(b\cdot\nabla_y)\subset\ker\mathcal{T}$. Conversely, for any $u\in\ker\mathcal{T}$ we have $\int_{\mathbb{R}^m}(b\cdot\nabla_y u)^2\,\mathrm{d}y=\int_{\mathbb{R}^m}u\mathcal{T}u\,\mathrm{d}y=0$ and therefore $u\in\ker(b\cdot\nabla_y)$.

Clearly Range $\mathcal{T}\subset$ Range $(b\cdot\nabla_y)=\ker\langle\cdot\rangle$. Consider now $w\in\ker\langle\cdot\rangle=$ Range $(b\cdot\nabla_y)$. By Proposition 3.2 there is $v\in\ker\langle\cdot\rangle\cap\mathrm{dom}(b\cdot\nabla_y)$ such that $b\cdot\nabla_y v=w$. Applying one more time Proposition 3.2, there is $u\in\ker\langle\cdot\rangle\cap\mathrm{dom}(b\cdot\nabla_y)$ such that $b\cdot\nabla_y u=v$. We deduce that $u\in\mathrm{dom}\mathcal{T}, w=\mathcal{T}(-u)$. Finally, for any $u\in\mathrm{dom}\mathcal{T}$ we apply twice the Poincaré inequality, taking into account that $\langle b\cdot\nabla_y u\rangle=0$

$$\|u-\langle u\rangle\,\|\leq C_P\|b\cdot\nabla_y u\|\leq C_P^2\|\mathcal{T}u\|.$$

$\square$

Remark 3.2 *The average along the flow of* b *can be defined in any Lebesgue space* $L^q(\mathbb{R}^m)$, $q\in[1,+\infty]$. *We refer to [4] for a complete presentation of these results.*

3.1 Average and first order differential operators

We are looking for first order derivations commuting with the average operator. Recall that the commutator $[\xi\cdot\nabla_y,\eta\cdot\nabla_y]$ between two first order differential operators is still a first order differential operator, whose vector field, denoted by $[\xi,\eta]$, is given by the Poisson bracket between ξ and η

$$[\xi\cdot\nabla_y,\eta\cdot\nabla_y]:=\xi\cdot\nabla_y(\eta\cdot\nabla_y)-\eta\cdot\nabla_y(\xi\cdot\nabla_y)=[\xi,\eta]\cdot\nabla_y$$

where $[\xi,\eta]=(\xi\cdot\nabla_y)\eta-(\eta\cdot\nabla_y)\xi$. The two vector fields ξ and η are said in involution iff their Poisson bracket vanishes.

Assume that $c(y)$ is a smooth vector field, satisfying $c(Y(s;y))=\partial_y Y(s;y)c(y), s\in\mathbb{R}, y\in\mathbb{R}^m$, where Y is the flow of b (not necessarily divergence free here). Taking the derivative with respect to s at $s=0$ yields $(b\cdot\nabla_y)c=\partial_y b\ c(y)$, saying that $[b,c]=0$. Actually the converse implication holds true and we obtain the following characterization for vector fields in involution, which is valid in distributions as well (see Appendix A for proof details).

Proposition 3.4 *Consider* $b\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m)$ *(not necessarily divergence free), with linear growth and* $c\in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. *Then* $(b\cdot\nabla_y)c-\partial_y b\ c=0$ *in* $\mathcal{D}'(\mathbb{R}^m)$ *iff*

$$c(Y(s;y))=\partial_y Y(s;y)c(y),\quad s\in\mathbb{R},\quad y\in\mathbb{R}^m. \tag{30}$$

We establish also weak formulations characterizing the involution between two fields, in distribution sense (see Appendix A for the proof). The notation w_s stands for $w\circ Y(s;\cdot)$.

Proposition 3.5 *Consider $b \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^m)$, with linear growth and zero divergence and $c \in L^1_{\text{loc}}(\mathbb{R}^m)$. Then the following statements are equivalent*

1.

$$[b, c] = 0 \textit{ in } \mathcal{D}'(\mathbb{R}^m)$$

2.

$$\int_{\mathbb{R}^m} (c \cdot \nabla_y u) v_{-s} \, \mathrm{d}y = \int_{\mathbb{R}^m} (c \cdot \nabla_y u_s) v \, \mathrm{d}y, \quad \forall \, u, v \in C^1_c(\mathbb{R}^m) \tag{31}$$

3.

$$\int_{\mathbb{R}^m} c \cdot \nabla_y u \; b \cdot \nabla_y v \, \mathrm{d}y + \int_{\mathbb{R}^m} c \cdot \nabla_y (b \cdot \nabla_y u) v \, \mathrm{d}y = 0, \quad \forall \, u \in C^2_c(\mathbb{R}^m), \;\; v \in C^1_c(\mathbb{R}^m). \tag{32}$$

Remark 3.3 *If $[b, c] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$, applying* (31) *with $v = 1$ on the support of u_s (and therefore $v_{-s} = 1$ on the support of u) yields*

$$\int_{\mathbb{R}^m} c \cdot \nabla_y u \, \mathrm{d}y = \int_{\mathbb{R}^m} c \cdot \nabla_y u_s \, \mathrm{d}y, \quad u \in C^1_c(\mathbb{R}^m)$$

saying that $\mathrm{div}_y c$ *is constant along the flow of b (in $\mathcal{D}'(\mathbb{R}^m)$).*

We claim that for vector fields c in involution with b, the derivation $c \cdot \nabla_y$ commutes with the average operator. This comes easily by the commutation property between the flows of b and c. Indeed, if $Z(h; y)$ stands for the flow of the vector field c (assumed smooth for the moment) we have, thanks to the involution property between b and c

$$Z(h; \cdot) \circ Y(s; \cdot) = Y(s; \cdot) \circ Z(h; \cdot), \quad s, h \in \mathbb{R}.$$

In order to establish $\langle c \cdot \nabla_y u \rangle = c \cdot \nabla_y \langle u \rangle$ it is enough to prove that $\langle u \circ Z(h; \cdot) \rangle = \langle u \rangle \circ Z(h; \cdot)$ for any $h \in \mathbb{R}$. At least formally we can write

$$\begin{aligned}
\langle u \circ Z(h; \cdot) \rangle &= \lim_{T \to +\infty} \frac{1}{T} \int_0^T u \circ Z(h; \cdot) \circ Y(s; \cdot) \, \mathrm{d}s \\
&= \lim_{T \to +\infty} \frac{1}{T} \int_0^T u \circ Y(s; \cdot) \circ Z(h; \cdot) \, \mathrm{d}s \\
&= \left(\lim_{T \to +\infty} \frac{1}{T} \int_0^T u \circ Y(s; \cdot) \, \mathrm{d}s \right) \circ Z(h; \cdot) \\
&= \langle u \rangle \circ Z(h; \cdot).
\end{aligned}$$

The rigorous statement and proof of this result follow below.

Proposition 3.6 *Consider a vector field $c \in L^1_{\text{loc}}(\mathbb{R}^m)$ with bounded divergence, in involution with b, that is $[b, c] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. Then the operators $u \to c \cdot \nabla_y u$, $u \to \mathrm{div}_y(uc)$ commute* *with the average operator i.e., for any* $u \in \mathrm{dom}(c \cdot \nabla_y) = \mathrm{dom}(\mathrm{div}_y(\cdot\ c))$ *we have* $\langle u \rangle \in \mathrm{dom}(c \cdot \nabla_y) = \mathrm{dom}(\mathrm{div}_y(\cdot\ c))$ *and*

$$\langle c \cdot \nabla_y u \rangle = c \cdot \nabla_y \langle u \rangle\,, \quad \langle \mathrm{div}_y(uc) \rangle = \mathrm{div}_y(\langle u \rangle\, c).$$

Proof. Consider $u \in \mathrm{dom}(c \cdot \nabla_y), s \in \mathbb{R}$ and $\varphi \in C^1_c(\mathbb{R}^m)$. We have

$$\begin{aligned} \int_{\mathbb{R}^m} u_s c \cdot \nabla_y \varphi \,\mathrm{d}y &= \int_{\mathbb{R}^m} u (c \cdot \nabla_y \varphi)_{-s} \,\mathrm{d}y \\ &= \int_{\mathbb{R}^m} u (c \cdot \nabla_y) \varphi_{-s} \,\mathrm{d}y \\ &= - \int_{\mathbb{R}^m} \mathrm{div}_y(uc) \varphi_{-s} \,\mathrm{d}y \\ &= - \int_{\mathbb{R}^m} (\mathrm{div}_y(uc))_s \varphi(y) \,\mathrm{d}y \end{aligned} \tag{33}$$

saying that $u_s \in \mathrm{dom}(c \cdot \nabla_y) = \mathrm{dom}(\mathrm{div}_y(\cdot\ c))$ and $\mathrm{div}_y(u_s c) = (\mathrm{div}_y(uc))_s$. We deduce $c \cdot \nabla_y u_s = (c \cdot \nabla_y u)_s$ cf. Remark 3.3. Integrating (33) with respect to s between 0 and $T > 0$ one gets

$$\begin{aligned} \int_{\mathbb{R}^m} \frac{1}{T} \int_0^T u_s \mathrm{d}s\ c \cdot \nabla_y \varphi \,\mathrm{d}y &= \frac{1}{T} \int_0^T \int_{\mathbb{R}^m} u_s c \cdot \nabla_y \varphi \,\mathrm{d}y\mathrm{d}s \\ &= -\frac{1}{T} \int_0^T \int_{\mathbb{R}^m} (\mathrm{div}_y(uc))_s \varphi(y) \,\mathrm{d}y\mathrm{d}s \\ &= - \int_{\mathbb{R}^m} \frac{1}{T} \int_0^T (\mathrm{div}_y(uc))_s \mathrm{d}s\ \varphi(y) \,\mathrm{d}y. \end{aligned}$$

By Proposition 3.1 we know that $\frac{1}{T}\int_0^T u_s \mathrm{d}s \to \langle u \rangle$ and $\frac{1}{T}\int_0^T (\mathrm{div}_y(uc))_s \mathrm{d}s \to \langle \mathrm{div}_y(uc) \rangle$ strongly in $L^2(\mathbb{R}^m)$, when $T \to +\infty$, and thus we obtain

$$\int_{\mathbb{R}^m} \langle u \rangle\, c \cdot \nabla_y \varphi \,\mathrm{d}y = - \int_{\mathbb{R}^m} \langle \mathrm{div}_y(uc) \rangle\, \varphi(y) \,\mathrm{d}y$$

saying that $\langle u \rangle \in \mathrm{dom}(c \cdot \nabla_y)$ and $\mathrm{div}_y(\langle u \rangle\, c) = \langle \mathrm{div}_y(uc) \rangle$, $c \cdot \nabla_y \langle u \rangle = \langle c \cdot \nabla_y u \rangle$. $\square$

3.2 Average and second order differential operators

We investigate the second order differential operators $-\mathrm{div}_y(A(y)\nabla_y)$ commuting with the average operator along the flow of b, where $A(y)$ is a smooth field of symmetric matrices. Such second order operators leave invariant $\ker(b \cdot \nabla_y)$. Indeed, for any $u \in \mathrm{dom}(-\mathrm{div}_y(A(y)\nabla_y)) \cap \ker(b \cdot \nabla_y)$ we have

$$-\mathrm{div}_y(A(y)\nabla_y u) = -\mathrm{div}_y(A(y)\,\langle u \rangle) = \langle -\mathrm{div}_y(A(y)\nabla_y u) \rangle \in \ker(b \cdot \nabla_y).$$

For this reason it is worth considering the operators $-\mathrm{div}_y(A(y)\nabla_y)$ commuting with $b \cdot \nabla_y$. A straightforward computation shows that

Proposition 3.7 *Consider a divergence free vector field $b \in W^{2,\infty}(\mathbb{R}^m)$ and a matrix field $A \in W^{2,\infty}(\mathbb{R}^m)$. The commutator between $b \cdot \nabla_y$ and $-\mathrm{div}_y(A(y)\nabla_y)$ is still a second order differential operator*

$$[b \cdot \nabla_y, -\mathrm{div}_y(A\nabla_y)] = -\mathrm{div}_y([b, A]\nabla_y)$$

whose matrix field, denoted by $[b, A]$, is given by

$$[b, A] = (b \cdot \nabla_y)A - \partial_y b A(y) - A(y)\, {}^t\partial_y b, \quad y \in \mathbb{R}^m.$$

Remark 3.4 *We have the formula ${}^t[b, A] = [b, {}^tA]$. In particular if $A(y)$ is a field of symmetric (resp. anti-symmetric) matrices, the field $[b, A]$ has also symmetric (resp. anti-symmetric) matrices.*

As for vector fields in involution, we have the following characterization (see Appendix A for proof details).

Proposition 3.8 *Consider $b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m)$ (not necessarily divergence free) with linear growth and $A(y) \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. Then $[b, A] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ iff*

$$A(Y(s; y)) = \partial_y Y(s; y)A(y)\, {}^t\partial_y Y(s; y), \quad s \in \mathbb{R}, \quad y \in \mathbb{R}^m. \tag{34}$$

For fields of symmetric matrices we have the weak characterization (see Appendix A for the proof).

Proposition 3.9 *Consider $b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m)$ with linear growth, zero divergence and $A \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ a field of symmetric matrices. Then the following statements are equivalent*

1.

$$[b, A] = 0 \ \ in \ \ \mathcal{D}'(\mathbb{R}^m).$$

2.

$$\int_{\mathbb{R}^m} A(y)\nabla_y u_s \cdot \nabla_y v_s \,\mathrm{d}y = \int_{\mathbb{R}^m} A(y)\nabla_y u \cdot \nabla_y v \,\mathrm{d}y$$

for any $s \in \mathbb{R}$, $u, v \in C^1_c(\mathbb{R}^m)$.

3.

$$\int_{\mathbb{R}^m} A(y)\nabla_y(b \cdot \nabla_y u) \cdot \nabla_y v \,\mathrm{d}y + \int_{\mathbb{R}^m} A(y)\nabla_y u \cdot \nabla_y(b \cdot \nabla_y v) \,\mathrm{d}y = 0$$

for any $u, v \in C^2_c(\mathbb{R}^m)$.

We consider the (formal) adjoint of the linear operator $A \to [b, A]$, with respect to the scalar product $(U, V) = \int_{\mathbb{R}^m} U(y) : V(y) \,\mathrm{d}y$, given by

$$Q \to -(b \cdot \nabla_y)Q - {}^t\partial_y b Q(y) - Q(y)\partial_y b$$

when $\text{div}_y b = 0$. The following characterization comes easily and the proof is left to the reader.

Proposition 3.10 *Consider $b \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^m)$, with linear growth and $Q \in L^1_{\text{loc}}(\mathbb{R}^m)$. Then $-(b \cdot \nabla_y)Q - {}^t\partial_y b Q(y) - Q(y)\partial_y b = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ iff*

$$Q(Y(s;y)) = {}^t\partial_y Y^{-1}(s;y) Q(y) \partial_y Y^{-1}(s;y), \;\; s \in \mathbb{R}, \;\; y \in \mathbb{R}^m. \tag{35}$$

Remark 3.5 *If $Q(y)$ satisfies* (35) *and is invertible for any $y \in \mathbb{R}^m$ with $Q^{-1} \in L^1_{\text{loc}}(\mathbb{R}^m)$, then $Q^{-1}(Y(s;y)) = \partial_y Y(s;y) Q^{-1}(y) {}^t\partial_y Y(s;y)$, $s \in \mathbb{R}, y \in \mathbb{R}^m$ and therefore $[b, Q^{-1}] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. If $P(y)$ satisfies* (34) *and is invertible for any $y \in \mathbb{R}^m$, then*

$$P^{-1}(Y(s;y)) = {}^t\partial_y Y^{-1}(s;y) P^{-1}(y) \partial_y Y^{-1}(s;y), \;\; s \in \mathbb{R}, \;\; y \in \mathbb{R}^m$$

and therefore $-(b \cdot \nabla_y)P^{-1} - {}^t\partial_y b P^{-1}(y) - P^{-1}(y)\partial_y b = 0$ in $\mathcal{D}'(\mathbb{R}^m)$.

As for vector fields in involution, the matrix fields in involution with b generate second order differential operators commuting with the average operator.

Proposition 3.11 *Consider a matrix field $A \in L^1_{\text{loc}}(\mathbb{R}^m)$ such that $\text{div}_y A \in L^1_{\text{loc}}(\mathbb{R}^m)$ and $[b, A] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. Therefore the operator $u \to -\text{div}_y(A\nabla_y u)$ commutes with the average operator i.e., for any $u \in \text{dom}(-\text{div}_y(A\nabla_y))$ we have $\langle u \rangle \in \text{dom}(-\text{div}_y(A\nabla_y))$ and*

$$-\langle \text{div}_y(A\nabla_y u) \rangle = -\text{div}_y(A\nabla_y \langle u \rangle).$$

Proof. Consider $u \in \text{dom}(-\text{div}_y(A\nabla_y)) = \{w \in L^2(\mathbb{R}^m) : -\text{div}_y(A\nabla_y w) \in L^2(\mathbb{R}^m)\}$. For any $s \in \mathbb{R}, \varphi \in C^2_c(\mathbb{R}^m)$ we have

$$-\int_{\mathbb{R}^m} u_s \, \text{div}_y({}^tA\nabla_y \varphi) \, \mathrm{d}y = -\int_{\mathbb{R}^m} u \, (\text{div}_y({}^tA\nabla_\varphi))_{-s} \, \mathrm{d}y. \tag{36}$$

By the implication 1. $\implies$ 2. of Proposition 3.9 (which does not require the symmetry of $A(y)$) we know that

$$\int_{\mathbb{R}^m} {}^tA\nabla_y \varphi \cdot \nabla_y \psi_s \, \mathrm{d}y = \int_{\mathbb{R}^m} {}^tA\nabla_y \varphi_{-s} \cdot \nabla_y \psi \, \mathrm{d}y$$

for any $\psi \in C^2_c(\mathbb{R}^m)$. We deduce that

$$-\int_{\mathbb{R}^m} \text{div}_y({}^tA\nabla_y \varphi)\psi_s \, \mathrm{d}y = -\int_{\mathbb{R}^m} \text{div}_y({}^tA\nabla_y \varphi_{-s})\psi \, \mathrm{d}y$$

and thus $(\text{div}_y({}^tA\nabla_y \varphi))_{-s} = \text{div}_y({}^tA\nabla_y \varphi_{-s})$. Combining with (36) yields

$$\begin{aligned} -\int_{\mathbb{R}^m} u_s \text{div}_y({}^tA\nabla_y \varphi) \, \mathrm{d}y &= -\textstyle\int_{\mathbb{R}^m} u \, \text{div}_y({}^tA\nabla_y \varphi_{-s}) \, \mathrm{d}y \\ &= -\textstyle\int_{\mathbb{R}^m} \text{div}_y(A\nabla_y u)\varphi_{-s} \, \mathrm{d}y \\ &= -\textstyle\int_{\mathbb{R}^m} (\text{div}_y(A\nabla_y u))_s \varphi(y) \, \mathrm{d}y \end{aligned} \tag{37}$$

saying that $u_s \in \mathrm{dom}(-\mathrm{div}_y(A\nabla_y))$ and

$$-\mathrm{div}_y(A\nabla_y u_s) = (-\mathrm{div}_y(A\nabla_y u))_s.$$

Integrating (37) with respect to s between 0 and T we obtain

$$\int_{\mathbb{R}^m} \frac{1}{T}\int_0^T u_s \,\mathrm{d}s\ \mathrm{div}_y({}^t A\nabla_y\varphi)\,\mathrm{d}y = \int_{\mathbb{R}^m} \frac{1}{T}\int_0^T (\mathrm{div}_y(A\nabla_y u))_s \,\mathrm{d}s\ \varphi(y)\,\mathrm{d}y.$$

Letting $T \to +\infty$ yields

$$\int_{\mathbb{R}^m} \langle u\rangle\, \mathrm{div}_y({}^t A\nabla_y\varphi)\,\mathrm{d}y = \int_{\mathbb{R}^m} \langle \mathrm{div}_y(A\nabla_y u)\rangle\, \varphi(y)\,\mathrm{d}y$$

and therefore $\langle u\rangle \in \mathrm{dom}(\mathrm{div}_y(A\nabla_y))$, $\mathrm{div}_y(A\nabla_y\langle u\rangle) = \langle \mathrm{div}_y(A\nabla_y u)\rangle$. □

3.3 The averaged diffusion matrix field

We are looking for the limit, when $\varepsilon \to 0$, of (1), (2). We expect that the limit $u = \lim_{\varepsilon\searrow 0} u^\varepsilon$ satisfies (6), (7). By (6) we deduce that at any time $t \in \mathbb{R}_+$, $u(t,\cdot) \in \ker(b\cdot\nabla_y)$. Observe also that $\mathrm{div}_y(b\otimes b\nabla_y u^1) = b\cdot\nabla_y(b\cdot\nabla_y u^1) \in \mathrm{Range}\,(b\cdot\nabla_y) \subset \ker\langle\cdot\rangle$ and therefore the closure for u comes by applying the average operator to (7) and by noticing that $\langle\partial_t u\rangle = \partial_t\langle u\rangle = \partial_t u$

$$\partial_t u - \langle \mathrm{div}_y(D\nabla_y u)\rangle = 0,\ \ t\in\mathbb{R}_+,\ \ y\in\mathbb{R}^m. \tag{38}$$

At least when $[b,D]=0$, we know by Proposition 3.11 that

$$\langle \mathrm{div}_y(D\nabla_y u)\rangle = \mathrm{div}_y(D\nabla_y\langle u\rangle) = \mathrm{div}_y(D\nabla_y u)$$

and (38) reduces to the diffusion equation associated to the matrix field $D(y)$. Nevertheless, even if $[b,D]\neq 0$, (38) behaves like a diffusion equation. More exactly the $L^2(\mathbb{R}^m)$ norm of the solution decreases with a rate proportional to the $L^2(\mathbb{R}^m)$ norm of its gradient under the hypothesis (3)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m}(u(t,y))^2\,\mathrm{d}y &= \int_{\mathbb{R}^m}\langle \mathrm{div}_y(D\nabla_y u)\rangle\, u(t,y)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^m}\mathrm{div}_y(D\nabla_y u)u\,\mathrm{d}y\\ &= -\int_{\mathbb{R}^m} D\nabla_y u\cdot\nabla_y u\,\mathrm{d}y\\ &= -\int_{\mathbb{R}^m}(D+b\otimes b) : \nabla_y u\otimes\nabla_y u\,\mathrm{d}y\\ &\le -d\int_{\mathbb{R}^m}|\nabla_y u(t,y)|^2\,\mathrm{d}y.\end{aligned}$$

We expect that, under appropriate hypotheses, (38) coincides with a diffusion equation, corresponding to some *averaged* matrix field $\mathcal{D}$, that is

$$\exists\, \mathcal{D}(y)\ :\ [b, \mathcal{D}] = 0 \ \text{ and } \ \langle -\mathrm{div}_y(D\nabla_y u)\rangle = -\mathrm{div}_y(\mathcal{D}\nabla_y u),\ \ \forall\, u \in \ker(b\cdot\nabla_y). \tag{39}$$

It is easily seen that in this case the limit model (38) reduces to

$$\partial_t u - \mathrm{div}_y(\mathcal{D}\nabla_y u) = 0,\ \ t\in\mathbb{R}_+,\ \ y\in\mathbb{R}^m.$$

In this section we identify sufficient conditions which guarantee the existence of the matrix field $\mathcal{D}$. We will see that it appears as the long time limit of the solution of another parabolic type problem, whose initial data is D, and thus as the orthogonal projection of the field $D(y)$ (with respect to some scalar product to be defined) on a subset of $\{A \in L^1_{\mathrm{loc}}(\mathbb{R}^m) : [b, A] = 0 \text{ in } \mathcal{D}'(\mathbb{R}^m)\}$. We assume that (18) holds true. We introduce the set

$$H_Q = \{A = A(y)\ :\ \int_{\mathbb{R}^m} Q(y)A(y) : A(y)Q(y)\,\mathrm{d}y < +\infty\}$$

where $Q = P^{-1}$ and the bilinear application

$$(\cdot,\cdot)_Q : H_Q\times H_Q \to \mathbb{R},\ \ (A,B)_Q = \int_{\mathbb{R}^m} Q(y)A(y) : B(y)Q(y)\,\mathrm{d}y$$

which is symmetric and positive definite. Indeed, for any $A\in H_Q$ we have

$$(A,A)_Q = \int_{\mathbb{R}^m} Q^{1/2}AQ^{1/2} : Q^{1/2}AQ^{1/2}\,\mathrm{d}y \ge 0$$

with equality iff $Q^{1/2}AQ^{1/2} = 0$ and thus iff $A = 0$. The set H_Q endowed with the scalar product $(\cdot,\cdot)_Q$ becomes a Hilbert space, whose norm is denoted by $|A|_Q = (A,A)_Q^{1/2}$, $A\in H_Q$. Observe that $H_Q \subset \{A(y) : A \in L^1_{\mathrm{loc}}(\mathbb{R}^m)\}$. Indeed, if for any matrix M the notation $|M|$ stands for the norm subordonated to the euclidian norm of $\mathbb{R}^m$

$$|M| = \sup_{\xi\in\mathbb{R}^m\setminus\{0\}} \frac{|M\xi|}{|\xi|} \le (M:M)^{1/2}$$

we have for a.a. $y\in\mathbb{R}^m$

$$\begin{aligned}
|A(y)| &= \sup_{\xi,\eta\neq 0} \frac{A(y)\xi\cdot\eta}{|\xi|\,|\eta|} \qquad (40)\\
&= \sup_{\xi,\eta\neq 0} \frac{Q^{1/2}AQ^{1/2}P^{1/2}\xi\cdot P^{1/2}\eta}{|P^{1/2}\xi|\,|P^{1/2}\eta|}\ \frac{|P^{1/2}\xi|}{|\xi|}\ \frac{|P^{1/2}\eta|}{|\eta|}\\
&\le |Q^{1/2}AQ^{1/2}|\,|P^{1/2}|^2\\
&\le (Q^{1/2}AQ^{1/2} : Q^{1/2}AQ^{1/2})^{1/2}\,|P|.
\end{aligned}$$

We deduce that for any $R>0$

$$\int_{B_R} |A(y)|\,\mathrm{d}y \le \int_{B_R} (Q^{1/2}AQ^{1/2} : Q^{1/2}AQ^{1/2})^{1/2}\,|P|\,\mathrm{d}y \le (A,A)_Q^{1/2}\left(\int_{B_R}|P(y)|^2\,\mathrm{d}y\right)^{1/2}.$$

Remark 3.6 *We know by Remark 3.5 that* $Q_s = {}^t\partial_y Y^{-1}(s;y)Q(y)\partial_y Y^{-1}(s;y)$ *which writes* ${}^t\mathcal{O}(s;y)\mathcal{O}(s;y) = I$ *where* $\mathcal{O}(s;y) = Q_s^{1/2}\partial_y Y(s;y)Q^{-1/2}$. *Therefore the matrix* $\mathcal{O}(s;y)$ *are orthogonal and we have*

$$Q_s^{1/2}\partial_y Y(s;y)Q^{-1/2} = \mathcal{O}(s;y) = {}^t\mathcal{O}^{-1}(s;y) = Q_s^{-1/2}\, {}^t\partial_y Y^{-1} Q^{1/2} \tag{41}$$

$$Q^{-1/2}\, {}^t\partial_y Y(s;y)Q_s^{1/2} = {}^t\mathcal{O}(s;y) = \mathcal{O}^{-1}(s;y) = Q^{1/2}\partial_y Y^{-1} Q_s^{-1/2}. \tag{42}$$

As said before, the set of matrix fields in involution with the vector field b will play a crucial role. By Proposition 3.8 such matrix fields are characterized by

$$A(Y(s;y)) = \partial_y Y(s;y)A(y)\, {}^t\partial_y Y(s;y), \quad s \in \mathbb{R}, y \in \mathbb{R}^m$$

which also writes

$$\partial_y Y^{-1}(s;y)A(Y(s;y))\, {}^t\partial_y Y^{-1}(s;y) = A(y), \quad s \in \mathbb{R}, y \in \mathbb{R}^m.$$

It is natural to consider the family of linear applications $A \to \partial_y Y^{-1}(s;\cdot)A(Y(s;\cdot))\, {}^t\partial_y Y^{-1}(s;\cdot), s \in \mathbb{R}$, whose fixed points are exactly the matrix fields in involution with b.

Proposition 3.12 *The family of applications* $A \to G(s)A := \partial_y Y^{-1}(s;\cdot)A_s\, {}^t\partial_y Y^{-1}(s;\cdot)$ *is a* C^0*- group of unitary operators on* H_Q.

Proof. For any $A \in H_Q$ observe, thanks to (42), that

$$\begin{aligned}
\left|\partial_y Y^{-1}(s;\cdot)A_s{}^t\partial_y Y^{-1}(s;\cdot)\right|_Q^2 &= \int_{\mathbb{R}^m} Q^{1/2}\partial_y Y^{-1}A_s{}^t\partial_y Y^{-1}Q^{1/2} : Q^{1/2}\partial_y Y^{-1}A_s{}^t\partial_y Y^{-1}Q^{1/2}\,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} {}^t\mathcal{O}(s;y)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;y) : {}^t\mathcal{O}(s;y)Q_s^{1/2}A_sQ_s^{1/2}\mathcal{O}(s;y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} Q_s^{1/2}A_sQ_s^{1/2} : Q_s^{1/2}A_sQ_s^{1/2}\,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} Q^{1/2}AQ^{1/2} : Q^{1/2}AQ^{1/2}\,\mathrm{d}y \\
&= |A|_Q^2.
\end{aligned}$$

Clearly $G(0)A = A, A \in H_Q$ and for any $s, t \in \mathbb{R}$ we have

$$\begin{aligned}
G(s)G(t)A &= \partial_y Y^{-1}(s;\cdot)(G(t)A)_s{}^t\partial_y Y^{-1}(s;\cdot) \\
&= \partial_y Y^{-1}(s;\cdot)(\partial_y Y)^{-1}(t;Y(s;\cdot))(A_t)_s{}^t(\partial_y Y)^{-1}(t;Y(s;\cdot)){}^t\partial_y Y^{-1}(s;\cdot) \\
&= \partial_y Y^{-1}(t+s;\cdot)A_{t+s}{}^t\partial_y Y^{-1}(t+s;\cdot) = G(t+s)A, \quad A \in H_Q.
\end{aligned}$$

It remains to check the continuity of the group, i.e., $\lim_{s\to 0} G(s)A = A$ strongly in H_Q for any $A \in H_Q$. For any $s \in \mathbb{R}$ we have

$$|G(s)A - A|_Q^2 = |G(s)A|_Q^2 + |A|_Q^2 - 2(G(s)A, A)_Q = 2|A|_Q^2 - 2(G(s)A, A)_Q$$

and thus it is enough to prove that $\lim_{s\to 0} G(s)A = A$ weakly in H_Q. As $|G(s)| = 1$ for any $s \in \mathbb{R}$, we are done if we prove that $\lim_{s\to 0}(G(s)A, U)_Q = (A, U)_Q$ for any $U \in C_c^0(\mathbb{R}^m) \subset H_Q$. But it is easily seen that $\lim_{s\to 0} G(-s)U = U$ strongly in H_Q, for $U \in C_c^0(\mathbb{R}^m)$ and thus

$$\lim_{s\to 0}(G(s)A, U)_Q = \lim_{s\to 0}(A, G(-s)U)_Q = (A, U)_Q, \ \ U \in C_c^0(\mathbb{R}^m).$$

$\square$

We denote by L the infinitesimal generator of the group G

$$L : \mathrm{dom}(L) \subset H_Q \to H_Q, \ \ \mathrm{dom}L = \{A \in H_Q \ : \ \exists \lim_{s\to 0} \frac{G(s)A - A}{s} \ \text{ in } \ H_Q\}$$

and $L(A) = \lim_{s\to 0} \frac{G(s)A-A}{s}$ for any $A \in \mathrm{dom}(L)$. Notice that $C_c^1(\mathbb{R}^m) \subset \mathrm{dom}(L)$ and $L(A) = b \cdot \nabla_y A - \partial_y b A - A\, {}^t\partial_y b$, $A \in C_c^1(\mathbb{R}^m)$ (use the hypothesis $Q \in L^2_{\text{loc}}(\mathbb{R}^m)$ and the dominated convergence theorem). In other words $L(A)$ coincides with the bracket between b and A for any smooth matrix field A. As we will see in a moment (cf. statement 4 of Proposition 3.13), this equality holds in distribution sense for any matrix field $A \in \mathrm{dom}(L)$ and justifies the consideration of the C^0-group $\{G(s)\}_{s\in\mathbb{R}}$, leading to the infinitesimal generator L. Observe also that the group G commutes with transposition *i.e.* $G(s)\, {}^tA = {}^tG(s)A$, $s \in \mathbb{R}, A \in H_Q$ and for any $A \in \mathrm{dom}(L)$ we have ${}^tA \in \mathrm{dom}(L)$, $L({}^tA) = {}^tL(A)$. The main properties of the operator L are summarized below (when b is divergence free). In particular the operator L is skew-adjoint on H_Q, which is a direct consequence of our choice of Q in the weighted L^2 scalar product $(\cdot, \cdot)_Q$.

Proposition 3.13

1. *The domain of L is dense in H_Q and L is closed.*
2. *The matrix field $A \in H_Q$ belongs to* $\mathrm{dom}(L)$ *iff there is a constant $C > 0$ such that*

$$|G(s)A - A|_Q \le C|s|, \ \ s \in \mathbb{R}. \tag{43}$$

3. *The operator L is skew-adjoint.*
4. *For any $A \in \mathrm{dom}(L)$ we have*

$$-\mathrm{div}_y(L(A)\nabla_y) = b \cdot \nabla_y(-\mathrm{div}_y(A\nabla_y)) + \mathrm{div}_y(A\nabla_y(b \cdot \nabla_y)) \ \ \textit{in} \ \ \mathcal{D}'(\mathbb{R}^m)$$

that is

$$\int_{\mathbb{R}^m} L(A)\nabla_y u \cdot \nabla_y v \,\mathrm{d}y = -\int_{\mathbb{R}^m} A\nabla_y u \cdot \nabla_y(b \cdot \nabla_y v) \,\mathrm{d}y - \int_{\mathbb{R}^m} A\nabla_y(b \cdot \nabla_y u) \cdot \nabla_y v \,\mathrm{d}y$$

for any $u, v \in C_c^2(\mathbb{R}^m)$.

Proof. 1. The operator L is the infinitesimal generator of a C^0-group, and therefore $\mathrm{dom}(L)$ is dense and L is closed.

2. Assume that $A \in \mathrm{dom}(L)$. We know that $\frac{\mathrm{d}}{\mathrm{d}s} G(s)A = L(G(s)A) = G(s)L(A)$ and thus

$$|G(s)A - A|_Q = \left| \int_0^t G(\tau)L(A) \,\mathrm{d}\tau \right|_Q \leq \left| \int_0^s |G(\tau)L(A)|_Q \,\mathrm{d}\tau \right| = |s| \; |L(A)|_Q, \;\; s \in \mathbb{R}.$$

Conversely, assume that (43) holds true. Therefore we can extract a sequence $(s_k)_k$ converging to 0 such that

$$\lim_{k \to +\infty} \frac{G(s_k)A - A}{s_k} = V \;\text{ weakly in }\; H_Q.$$

For any $U \in \mathrm{dom}(L)$ we obtain

$$\left(\frac{G(s_k)A - A}{s_k}, U \right)_Q = \left(A, \frac{G(-s_k)U - U}{s_k} \right)_Q$$

and thus, letting $k \to +\infty$ yields

$$(V, U)_Q = -(A, L(U))_Q. \tag{44}$$

But since $U \in \mathrm{dom}(L)$, all the trajectory $\{G(\tau)U : \tau \in \mathbb{R}\}$ is contained in $\mathrm{dom}(L)$ and $G(-s_k)U = U + \int_0^{-s_k} L(G(\tau)U)\mathrm{d}\tau$. We deduce

$$\begin{aligned}
(G(s_k)A - A, U)_Q &= \left(A, \int_0^{-s_k} L(G(\tau)U) \,\mathrm{d}\tau \right) \\
&= \int_0^{-s_k} (A, L(G(\tau)U))_Q \,\mathrm{d}\tau \\
&= -\int_0^{-s_k} (V, G(\tau)U)_Q \,\mathrm{d}\tau \\
&= -\left(V, \int_0^{-s_k} G(\tau)U \,\mathrm{d}\tau \right)_Q .
\end{aligned}$$

Taking into account that $\left| \int_0^{-s_k} G(\tau)U \mathrm{d}\tau \right|_Q \leq |s_k| \; |U|_Q$ we obtain

$$\left| \left(\frac{G(s_k)A - A}{s_k}, U \right)_Q \right| \leq |V|_Q |U|_Q, \;\; U \in \mathrm{dom}(L)$$

and thus, by the density of $\mathrm{dom}(L)$ in H_Q one gets

$$\left| \frac{G(s_k)A - A}{s_k} \right|_Q \leq |V|_Q, \;\; k \in \mathbb{N}.$$

Since V is the weak limit in H_Q of $\left(\frac{G(s_k)A - A}{s_k} \right)_k$, we deduce that $\lim_{k \to +\infty} \frac{G(s_k)A - A}{s_k} = V$ strongly in H_Q. As the limit V is uniquely determined by (44), all the family $\left(\frac{G(s)A - A}{s} \right)_s$ converges strongly , when $s \to 0$, towards V in H_Q and thus $A \in \mathrm{dom}(L)$.

3. For any $U, V \in \mathrm{dom}(L)$ we can write

$$(G(s)U - U, V)_Q + (U, V - G(-s)V)_Q = 0, \;\; s \in \mathbb{R}.$$

Taking into account that

$$\lim_{s\to 0}\frac{G(s)U-U}{s}=L(U),\ \ \lim_{s\to 0}\frac{V-G(-s)V}{s}=L(V)$$

we obtain $(L(U),V)_Q+(U,L(V))_Q=0$ saying that $V\in \mathrm{dom}(L^\star)$ and $L^\star(V)=-L(V)$. Therefore $L\subset(-L^\star)$. It remains to establish the converse inclusion. Let $V\in\mathrm{dom}(L^\star)$, *i.e.*, $\exists C>0$ such that

$$|(L(U),V)_Q|\le C|U|_Q,\ \ U\in\mathrm{dom}(L).$$

For any $s\in\mathbb{R}$, $U\in\mathrm{dom}(L)$ we have

$$(G(s)V-V,U)_Q=(V,G(-s)U-U)_Q=(V,\int_0^{-s}LG(\tau)U\,\mathrm{d}\tau)_Q=\int_0^{-s}(V,LG(\tau)U)_Q\,\mathrm{d}\tau$$

implying

$$|(G(s)V-V,U)_Q|\le C|s|\,|U|_Q,\ \ s\in\mathbb{R}.$$

Therefore $|G(s)V-V|_Q\le C|s|, s\in\mathbb{R}$ and by the previous statement $V\in\mathrm{dom}(L)$. Finally $\mathrm{dom}(L)=\mathrm{dom}(L^\star)$ and $L^\star(V)=-L(V), V\in\mathrm{dom}(L)=\mathrm{dom}(L^\star)$.

4. As L is skew-adjoint, we obtain

$$-\int_{\mathbb{R}^m}L(A)\nabla_y u\cdot\nabla_y v\,\mathrm{d}y=-(L(A),Q^{-1}\nabla_y v\otimes\nabla_y uQ^{-1}\,)_Q=(A,L(Q^{-1}\nabla_y v\otimes\nabla_y uQ^{-1})\,)_Q.$$

Recall that $P=Q^{-1}$ satisfies $L(P)=0$, that is, $G(s)P=P, s\in\mathbb{R}$ and thus

$$\begin{aligned}L(Q^{-1}\nabla_y v\otimes\nabla_y uQ^{-1})&=\lim_{s\to 0}\frac{G(s)P\nabla_y v\otimes\nabla_y uP-P\nabla_y v\otimes\nabla_y uP}{s}\\&=\lim_{s\to 0}\frac{\partial_y Y^{-1}(s;\cdot)P_s(\nabla_y v)_s\otimes(\nabla_y u)_sP_s{}^t\partial_y Y^{-1}(s;\cdot)-P\nabla_y v\otimes\nabla_y uP}{s}\\&=\lim_{s\to 0}\frac{P^t\partial_y Y(s;\cdot)(\nabla_y v)_s\otimes(\nabla_y u)_s\partial_y Y(s;\cdot)P-P\nabla_y v\otimes\nabla_y uP}{s}\\&=\lim_{s\to 0}\frac{P\nabla_y v_s\otimes\nabla_y u_sP-P\nabla_y v\otimes\nabla_y uP}{s}\\&=P\nabla_y(b\cdot\nabla_y v)\otimes\nabla_y uP+P\nabla_y v\otimes\nabla_y(b\cdot\nabla_y u)P.\end{aligned}$$

Finally one gets

$$\begin{aligned}-\int_{\mathbb{R}^m}L(A)\nabla_y u\cdot\nabla_y v\,\mathrm{d}y&=(A,P\nabla_y(b\cdot\nabla_y v)\otimes\nabla_y uP)+P\nabla_y v\otimes\nabla_y(b\cdot\nabla_y u)P)_Q\\&=\int_{\mathbb{R}^m}A\nabla_y u\cdot\nabla_y(b\cdot\nabla_y v)\,\mathrm{d}y+\int_{\mathbb{R}^m}A\nabla_y(b\cdot\nabla_y u)\cdot\nabla_y v\,\mathrm{d}y.\end{aligned}$$

$\square$

We claim that $\mathrm{dom}(L)$ is left invariant by some special (weighted with respect to the matrix field Q) positive/negative part functions. The notations $A^{\pm}$ stand for the usual positive/negative parts of a symmetric matrix A

$$A^{\pm} = S\Lambda^{\pm}\,{}^tS, \quad A = S\Lambda\,{}^tS$$

where $\Lambda, \Lambda^{\pm}$ are the diagonal matrices containing the eigenvalues of A and the positive/negative parts of these eigenvalues respectively, and S is the orthogonal matrix whose columns contain a orthonormal basis of eigenvectors for A. Notice that

$$A^+ : A^- = 0, \quad A^+ - A^- = A, \quad A^+ : A^+ + A^- : A^- = A : A.$$

We introduce also the weighted positive/negative part functions which associate to any field of symmetric matrices $A(y)$ the fields of symmetric matrices $A^{Q\pm}(y)$ given by

$$Q^{1/2} A^{Q\pm}\, Q^{1/2} = (Q^{1/2} A Q^{1/2})^{\pm}.$$

Observe that $A^{Q+} - A^{Q-} = A$. We intend to study the trajectories of (19) and in particular, we want to prove that for any initial positive matrix field, the corresponding trajectory remains positive. We need to analyze how the infinitesimal generator L behaves when the matrix field splits into positive/negative parts. It happens that L separates the weighted positive/negative parts, which justifies their definitions. The detailed proof of this result is technical and can be found in Appendix A.

Proposition 3.14
1. *The applications $A \to A^{Q\pm}$ leave invariant the subset $\{A \in \mathrm{dom}(L) : {}^tA = A\}$.*
2. *For any $A \in \mathrm{dom}(L), {}^tA = A$ we have*

$$(A^{Q+}, A^{Q-})_Q = 0, \quad (L(A^{Q+}), L(A^{Q-}))_Q \leq 0.$$

We want to solve the problem (19), (20) by using variational methods. We introduce the space $V_Q = \mathrm{dom}(L) \subset H_Q$ endowed with the scalar product

$$((A, B))_Q = (A, B)_Q + (L(A), L(B))_Q, \quad A, B \in V_Q.$$

Clearly $(V_Q, ((\cdot,\cdot))_Q)$ is a Hilbert space (use the fact that L is closed) and the inclusion $V_Q \subset H_Q$ is continuous, with dense image. The notation $\|\cdot\|_Q$ stands for the norm associated to the scalar product $((\cdot,\cdot))_Q$

$$\|A\|_Q^2 = ((A, A))_Q = (A, A)_Q + (L(A), L(A))_Q = |A|_Q^2 + |L(A)|_Q^2, \quad A \in V_Q.$$

We introduce the bilinear form $\sigma : V_Q \times V_Q \to \mathbb{R}$

$$\sigma(A, B) = (L(A), L(B))_Q, \quad A, B \in V_Q.$$

Notice that σ is coercive on V_Q with respect to H_Q

$$\sigma(A, A) + |A|_Q^2 = \|A\|_Q^2, \quad A \in V_Q.$$

By Theorems 1,2 pp. 620 [10] we deduce that for any $D \in H_Q$ there is a unique variational solution for (19), (20) that is $A \in C_b(\mathbb{R}_+; H_Q) \cap L^2(\mathbb{R}_+; V_Q)$, $\partial_t A \in L^2(\mathbb{R}_+; V_Q')$

$$A(0) = D, \quad \frac{\mathrm{d}}{\mathrm{d}t}(A(t), U)_Q + \sigma(A(t), U) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^m), \quad \forall\, U \in V_Q.$$

The long time limit of the solution of (19), (20) provides the averaged matrix field in (39). The key point is the skew-adjointness of the operator L. On the one hand it is easily seen by the energy dissipation that $L(A(\cdot)) \in L^2(\mathbb{R}_+; H_Q)$ and therefore $L(\lim_{t\to+\infty} A(t)) = \lim_{t\to+\infty} L(A(t)) = 0$. On the other hand the orthogonality between the kernel and range of L ensures that $(\lim_{t\to+\infty} A(t) - A(0)) \perp \ker L$, implying that $\lim_{t\to+\infty} A(t) = \mathrm{Proj}_{\ker L} A(0)$.

Proof. (of Theorem 2.1) The identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A(t)|_Q^2 + |L(A(t))|_Q^2 = 0, \quad t \in \mathbb{R}_+$$

gives the estimates

$$|A(t)|_Q \le |D|_Q, \quad t \in \mathbb{R}_+, \quad \int_0^{+\infty} |L(A(t))|_Q^2 \,\mathrm{d}t \le \frac{1}{2}|D|_Q^2.$$

Consider $(t_k)_k$ such that $t_k \to +\infty$ as $k \to +\infty$ and $(A(t_k))_k$ converges weakly towards some matrix field X in H_Q. For any $U \in \ker L$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(A(t), U)_Q = 0, \quad t \in \mathbb{R}_+$$

and therefore

$$(\mathrm{Proj}_{\ker L} D, U)_Q = (D, U)_Q = (A(0), U)_Q = (A(t_k), U)_Q = (X, U)_Q, \quad U \in \ker L. \tag{45}$$

Since $L(A) \in L^2(R_+; H_Q)$ we deduce that $\lim_{k\to+\infty} L(A(t_k)) = 0$ strongly in H_Q. For any $V \in V_Q$ we have

$$(X, L(V))_Q = \lim_{k\to+\infty}(A(t_k), L(V))_Q = -\lim_{k\to+\infty}(L(A(t_k)), V)_Q = 0.$$

We deduce that $X \in \mathrm{dom}(L^\star) = \mathrm{dom}(L)$ and $L(X) = 0$, which combined with (45) says that $X = \mathrm{Proj}_{\ker L} D$, or $X = \langle D \rangle_Q$. By the uniqueness of the limit we obtain $\lim_{t\to+\infty} A(t) =$

$\mathrm{Proj}_{\ker L} D$ weakly in H_Q. Assume now that ${}^t D = D$. As L commutes with transposition, we have $\partial_t {}^t A - L(L({}^t A)) = 0$, ${}^t A(0) = D$. By the uniqueness we obtain ${}^t A = A$ and thus

$${}^t \langle D \rangle_Q = {}^t(\mathrm{w} - \lim_{t \to +\infty} A(t)) = \mathrm{w} - \lim_{t \to +\infty} {}^t A(t) = \mathrm{w} - \lim_{t \to +\infty} A(t) = \langle D \rangle_Q \,.$$

Suppose that $D \geq 0$ and let us check that $\langle D \rangle_Q \geq 0$. By Proposition 3.14 we know that $A^{Q\pm}(t) \in V_Q$, $t \in \mathbb{R}_+$ and

$$(A^{Q+}(t), A^{Q-}(t))_Q = 0, \;\; (L(A^{Q+}(t)), L(A^{Q-}(t)))_Q \leq 0, \;\; t \in \mathbb{R}_+.$$

It is sufficient to consider the case of smooth solutions. Multiplying (19) by $-A^{Q-}(t)$ one gets

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A^{Q-}(t)|_Q^2 + |L(A^{Q-}(t)|_Q^2 &= (\partial_t A^{Q+}, A^{Q-}(t))_Q + (L(A^{Q+}(t)), L(A^{Q-}(t)))_Q \\ &\leq (\partial_t A^{Q+}, A^{Q-}(t))_Q. \end{aligned} \tag{46}$$

But for any $0 < h < t$ we have

$$(A^{Q+}(t) - A^{Q+}(t-h), A^{Q-}(t))_Q = -(A^{Q+}(t-h), A^{Q-}(t))_Q \leq 0$$

and therefore $(\partial_t A^{Q+}(t), A^{Q-}(t))_Q \leq 0$. Observe that $Q^{1/2} A^{Q-}(0) Q^{1/2} = (Q^{1/2} D Q^{1/2})^- = 0$, since $Q^{1/2} D Q^{1/2}$ is symmetric and positive. Thus $A^{Q-}(0) = 0$, and from (46) we obtain

$$\frac{1}{2}|A^{Q-}(t)|_Q^2 \leq \frac{1}{2}|A^{Q-}(0)|_Q^2 = 0$$

implying that $Q^{1/2} A(t) Q^{1/2} \geq 0$ and $A(t) \geq 0$, $t \in \mathbb{R}_+$. Take now any $U \in H_Q$, ${}^t U = U$, $U \geq 0$. By weak convergence we have

$$(\langle D \rangle_Q, U)_Q = \lim_{t \to +\infty} (A(t), U)_Q = \lim_{t \to +\infty} \int_{\mathbb{R}^m} Q^{1/2} A(t) Q^{1/2} : Q^{1/2} U Q^{1/2} \,\mathrm{d}y \geq 0$$

and thus $\langle D \rangle_Q \geq 0$. By construction $\langle D \rangle_Q = \mathrm{Proj}_{\ker L} D \in \ker L$. It remains to justify the second statement in (21), and (22). Take a bounded function $\varphi \in L^\infty(\mathbb{R}^m)$ which remains constant along the flow of b, that is $\varphi_s = \varphi$, $s \in \mathbb{R}$, and a smooth function $u \in C^1(\mathbb{R}^m)$ such that $u_s = u$, $s \in \mathbb{R}$ and

$$\int_{\mathbb{R}^m} (\nabla_y u \cdot Q^{-1} \nabla_y u)^2 \,\mathrm{d}y < +\infty.$$

We introduce the matrix field U given by

$$U(y) = \varphi(y) Q^{-1}(y) \; \nabla_y u \otimes \nabla_y u \; Q^{-1}(y), \;\; y \in \mathbb{R}^m.$$

On the one hand notice that $U \in H_Q$

$$
\begin{aligned}
|U|_Q^2 = \int_{\mathbb{R}^m} Q^{1/2} U Q^{1/2} : Q^{1/2} U Q^{1/2} \, \mathrm{d}y = \int_{\mathbb{R}^m} \varphi^2 |Q^{-1/2} \nabla_y u|^4 \, \mathrm{d}y \\
\leq \|\varphi\|_{L^\infty}^2 \int_{\mathbb{R}^m} (\nabla_y u \cdot Q^{-1} \nabla_y u)^2 \, \mathrm{d}y.
\end{aligned}
$$

On the other hand, we claim that $U \in \ker L$. Indeed, for any $s \in \mathbb{R}$ we have

$$
\nabla_y u = \nabla_y u_s = {}^t\partial_y Y(s;y)(\nabla_y u)_s
$$

and thus

$$
\begin{aligned}
Q_s U_s Q_s &= \varphi_s (\nabla_y u)_s \otimes (\nabla_y u)_s \\
&= \varphi \, ({}^t\partial_y Y^{-1} \nabla_y u) \otimes ({}^t\partial_y Y^{-1} \nabla_y u) \\
&= \varphi \, {}^t\partial_y Y^{-1} \, \nabla_y u \otimes \nabla_y u \, \partial_y Y^{-1} \\
&= {}^t\partial_y Y^{-1} Q U Q \partial_y Y^{-1}.
\end{aligned}
$$

Taking into account that $Q_s = {}^t\partial_y Y^{-1} Q \partial_y Y^{-1}$ we obtain

$$
{}^t\partial_y Y^{-1} Q \partial_y Y^{-1} U_s {}^t\partial_y Y^{-1} Q \partial_y Y^{-1} = {}^t\partial_y Y^{-1} Q U Q \partial_y Y^{-1}
$$

saying that $U_s(y) = \partial_y Y(s;y) U(y) {}^t\partial_y Y(s;y)$. As $\langle D \rangle_Q = \mathrm{Proj}_{\ker L} D$ one gets

$$
\begin{aligned}
0 = (D - \langle D \rangle_Q, U)_Q &= \int_{\mathbb{R}^m} (D - \langle D \rangle_Q) : QUQ \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} \varphi(y) (D - \langle D \rangle_Q) : \nabla_y u \otimes \nabla_y u \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} \varphi(y) \{ \nabla_y u \cdot D \nabla_y u - \nabla_y u \cdot \langle D \rangle_Q \nabla_y u \} \, \mathrm{d}y.
\end{aligned}
$$

In particular, taking $\varphi = 1$ we deduce that $\nabla_y u \cdot \langle D \rangle_Q \nabla_y u \in L^1(\mathbb{R}^m)$ and

$$
\int_{\mathbb{R}^m} \nabla_y u \cdot \langle D \rangle_Q \nabla_y u \, \mathrm{d}y = \int_{\mathbb{R}^m} \nabla_y u \cdot D \nabla_y u \, \mathrm{d}y = (D, Q^{-1} \, \nabla_y u \otimes \nabla_y u \, Q^{-1})_Q < +\infty
$$

since $D \in H_Q$, $Q^{-1} \nabla_y u \otimes \nabla_y u Q^{-1} \in H_Q$. Since $\langle D \rangle_Q \in \ker L$, the function $\nabla_y u \cdot \langle D \rangle_Q \nabla_y u$ remains constant along the flow of b

$$
(\nabla_y u)_s \cdot (\langle D \rangle_Q)_s (\nabla_y u)_s = (\nabla_y u)_s \cdot \partial_y Y(s;y) \langle D \rangle_Q \, {}^t\partial_y Y(s;y) (\nabla_y u)_s = \nabla_y u \cdot \langle D \rangle_Q \nabla_y u.
$$

Therefore the function $\nabla_y u \cdot \langle D \rangle_Q \nabla_y u$ verifies the variational formulation

$$
\nabla_y u \cdot \langle D \rangle_Q \nabla_y u \in L^1(\mathbb{R}^m), \;\; (\nabla_y u \cdot \langle D \rangle_Q \nabla_y u)_s = \nabla_y u \cdot \langle D \rangle_Q \nabla_y u, \;\; s \in \mathbb{R} \tag{47}
$$

and

$$
\int_{\mathbb{R}^m} \nabla_y u \cdot D \nabla_y u \, \varphi \, \mathrm{d}y = \int_{\mathbb{R}^m} \nabla_y u \cdot \langle D \rangle_Q \nabla_y u \, \varphi \, \mathrm{d}y, \;\; \forall \, \varphi \in L^\infty(\mathbb{R}^m), \; \varphi_s = \varphi, \; s \in \mathbb{R}. \tag{48}
$$

It is easily seen, thanks to the hypothesis $D \in L^\infty(\mathbb{R}^m)$, that (47), (48) also make sense for functions $u \in H^1(\mathbb{R}^m)$ such that $u_s = u$, $s \in \mathbb{R}$. We obtain

$$\nabla_y u \cdot \langle D \rangle_Q \nabla_y u = \langle \nabla_y u \cdot D \nabla_y u \rangle, \quad u \in H^1(\mathbb{R}^m), \quad u_s = u, \quad s \in \mathbb{R}$$

where the average operator in the right hand side should be understood in the $L^1(\mathbb{R}^m)$ setting cf. Remark 3.2. Moreover, if $u, v \in H^1(\mathbb{R}^m) \cap \ker(b \cdot \nabla_y)$ then $\langle D \rangle_Q^{1/2} \nabla_y u, \langle D \rangle_Q^{1/2} \nabla_y v$ belong to $L^2(\mathbb{R}^m)$ implying that $\nabla_y u \cdot \langle D \rangle_Q \nabla_y v \in L^1(\mathbb{R}^m)$. As before we check that $\nabla_y u \cdot \langle D \rangle_Q \nabla_y v$ remains constant along the flow of b and for any $\varphi \in L^\infty(\mathbb{R}^m)$, $\varphi_s = \varphi, s \in \mathbb{R}$ we can write

$$\begin{aligned}
2 \int_{\mathbb{R}^m} \nabla_y u \cdot D \nabla_y v \; \varphi \, \mathrm{d}y &= \int_{\mathbb{R}^m} \nabla_y (u+v) \cdot D \nabla_y (u+v) \; \varphi \, \mathrm{d}y \\
&- \int_{\mathbb{R}^m} \nabla_y u \cdot D \nabla_y u \; \varphi \, \mathrm{d}y - \int_{\mathbb{R}^m} \nabla_y v \cdot D \nabla_y v \; \varphi \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} \nabla_y (u+v) \cdot \langle D \rangle_Q \nabla_y (u+v) \; \varphi \, \mathrm{d}y \\
&- \int_{\mathbb{R}^m} \nabla_y u \cdot \langle D \rangle_Q \nabla_y u \; \varphi \, \mathrm{d}y - \int_{\mathbb{R}^m} \nabla_y v \cdot \langle D \rangle_Q \nabla_y v \; \varphi \, \mathrm{d}y \\
&= 2 \int_{\mathbb{R}^m} \nabla_y u \cdot \langle D \rangle_Q \nabla_y v \; \varphi \, \mathrm{d}y.
\end{aligned}$$

Finally one gets

$$\nabla_y u \cdot \langle D \rangle_Q \nabla_y v = \langle \nabla_y u \cdot D \nabla_y v \rangle, \quad u, v \in H^1(\mathbb{R}^m) \cap \ker(b \cdot \nabla_y).$$

Consider now the functions $u \in H^1(\mathbb{R}^m) \cap \ker(b \cdot \nabla_y)$ and $\psi \in C_c^2(\mathbb{R}^m)$. In order to prove that $\left\langle \nabla_y u \cdot \langle D \rangle_Q \nabla_y (b \cdot \nabla_y \psi) \right\rangle = 0$, where the average is understood in the $L^1(\mathbb{R}^m)$ setting, we need to check that

$$\int_{\mathbb{R}^m} \varphi(y) \; \nabla_y u \cdot \langle D \rangle_Q \nabla_y (b \cdot \nabla_y \psi) \, \mathrm{d}y = 0$$

for any $\varphi \in L^\infty(\mathbb{R}^m)$, $\varphi_s = \varphi, s \in \mathbb{R}$. Clearly $B(y) := \varphi(y) \langle D \rangle_Q (y) \in \ker L$ and therefore it is enough to prove that

$$\int_{\mathbb{R}^m} \nabla_y u \cdot B \nabla_y (b \cdot \nabla_y \psi) \, \mathrm{d}y = 0$$

for any $B \in \ker L$, which comes by the third statement of Proposition 3.9. $\square$

What is remarkable is that the averaged matrix field, which appears as the long time limit of a partial differential equation, can be computed point by point, under certain hypotheses, as explained below.

Remark 3.7 *Assume that there is u_0 satisfying $u_0(Y(s;y)) = u_0(y) + s$, $s \in \mathbb{R}, y \in \mathbb{R}^m$. Notice that u_0 could be multi-valued function (think to angular coordinates) but its gradient* *satisfies for a.a.* $y \in \mathbb{R}^m$ *and* $s \in \mathbb{R}$

$$\nabla_y u_0 = {}^t\partial_y Y(s;y)(\nabla_y u_0)_s$$

exactly as any function u *which remains constant along the flow of* b. *For this reason, the last equality in* (21) *holds true for any* $u, v \in H^1(\mathbb{R}^m) \cap \ker(b \cdot \nabla_y) \cup \{u_0\}$. *In the case when* $m-1$ *independent prime integrals of* b *are known i.e.,* $\exists u_1, ..., u_{m-1} \in H^1(\mathbb{R}^m) \cap \ker(b \cdot \nabla_y)$, *the average of the matrix field* D *comes by imposing*

$$\nabla_y u_i \cdot \langle D \rangle_Q \nabla_y u_j = \langle \nabla_y u_i \cdot D \nabla_y u_j \rangle, \ \ i, j \in \{0, ..., m-1\}.$$

4 First order approximation

We assume that the fields $D(y), b(y)$ are bounded on $\mathbb{R}^m$

$$D \in L^\infty(\mathbb{R}^m), \ \ b \in L^\infty(\mathbb{R}^m). \tag{49}$$

We solve (1), (2) by using variational methods. We consider the Hilbert spaces $V := H^1(\mathbb{R}^m) \subset H := L^2(\mathbb{R}^m)$ (the injection $V \subset H$ being continuous, with dense image) and the bilinear forms $a^\varepsilon : V \times V \to \mathbb{R}$ given by

$$a^\varepsilon(u,v) = \int_{\mathbb{R}^m} D(y) \nabla_y u \cdot \nabla_y v \ \mathrm{d}y + \frac{1}{\varepsilon} \int_{\mathbb{R}^m} (b \cdot \nabla_y u) \ (b \cdot \nabla_y v) \ \mathrm{d}y, \ \ u, v \in V.$$

Notice that for any $0 < \varepsilon \le 1$ and $v \in V$ we have

$$\begin{aligned} a^\varepsilon(v,v) + d|v|_H^2 &\ge \int_{\mathbb{R}^m} D(y)\nabla_y v \cdot \nabla_y v + (b \cdot \nabla_y v) \ (b \cdot \nabla_y v) \ \mathrm{d}y + d \int_{\mathbb{R}^m} (v(y))^2 \ \mathrm{d}y \\ &\ge d \int_{\mathbb{R}^m} |\nabla_y v|^2 \ \mathrm{d}y + d \int_{\mathbb{R}^m} (v(y))^2 \ \mathrm{d}y \\ &= d|v|_V^2 \end{aligned}$$

saying that a^ε is coercive on V with respect to H. By Theorems 1,2 pp. 620 [10] we deduce that for any $u_{\mathrm{in}}^\varepsilon \in H$, there is a unique variational solution for (1), (2), that is $u^\varepsilon \in C_b(\mathbb{R}_+; H) \cap L^2(\mathbb{R}_+; V)$ and

$$u^\varepsilon(0) = u_{\mathrm{in}}^\varepsilon, \ \ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} u^\varepsilon(t,y) v(y) \ \mathrm{d}y + a^\varepsilon(u^\varepsilon(t), v) = 0, \ \ \text{in } \mathcal{D}'(\mathbb{R}^m), \ \ \forall \, v \in V.$$

By standard arguments one gets

Proposition 4.1 *The solutions* $(u^\varepsilon)_\varepsilon$ *satisfy the estimates*

$$\|u^\varepsilon\|_{C_b(\mathbb{R}_+;H)} \le |u_{\mathrm{in}}^\varepsilon|_H, \ \ \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} |\nabla_y u^\varepsilon|^2 \ \mathrm{d}y\mathrm{d}t \le \frac{|u_{\mathrm{in}}^\varepsilon|_H^2}{2d}$$

and

$$\|b\cdot\nabla_y u^\varepsilon\|_{L^2(\mathbb{R}_+;H)} \le \left(\frac{\varepsilon}{2(1-\varepsilon)}\right)^{1/2} |u^\varepsilon_{\mathrm{in}}|_H,\ \ \varepsilon\in(0,1).$$

We are ready to prove the convergence of the family $(u^\varepsilon)_\varepsilon$, when $\varepsilon\searrow 0$, towards the solution of the heat equation associated to the averaged diffusion matrix field $\langle D\rangle_Q$. Roughly speaking, the first order approximation is given by

$$\partial_t u - \langle \mathrm{div}_y(D\nabla_y u)\rangle = 0$$

which becomes a parabolic equation, since the properties of the averaged diffusion matrix ensure that $\langle \mathrm{div}_y(D\nabla_y u)\rangle = \mathrm{div}_y(\langle D\rangle_Q \nabla_y u)$ fon any $u\in\ker(b\cdot\nabla_y)$.

Proof. (of Theorem 2.2) Based on the uniform estimates in Proposition 4.1, there is a sequence $(\varepsilon_k)_k$, converging to 0, such that

$$u^{\varepsilon_k}\rightharpoonup u\ \text{ weakly }\star\text{ in } L^\infty(\mathbb{R}_+;H),\ \ \nabla_y u^{\varepsilon_k}\rightharpoonup\nabla_y u\ \text{ weakly in }\ L^2(\mathbb{R}_+;H).$$

Using the weak formulation of (1) with test functions $\eta(t)\varphi(y)$, $\eta\in C^1_c(\mathbb{R}_+), \varphi\in C^1_c(\mathbb{R}^m)$ yields

$$\begin{aligned}-\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}\eta'(t)\varphi(y)u^{\varepsilon_k}(t,y)\,\mathrm{d}y\mathrm{d}t-\eta(0)\int_{\mathbb{R}^m}\varphi u^{\varepsilon_k}_{\mathrm{in}}\,\mathrm{d}y+\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}\eta\nabla_y u^{\varepsilon_k}\cdot D\nabla_y\varphi\,\mathrm{d}y\mathrm{d}t\\ =-\frac{1}{\varepsilon_k}\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}\eta(t)(b\cdot\nabla_y u^{\varepsilon_k})(b\cdot\nabla_y\varphi)\,\mathrm{d}y\mathrm{d}t.\end{aligned}\tag{50}$$

Multiplying by ε_k and letting $k\to+\infty$, it is easily seen that

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}\eta(b\cdot\nabla_y u)\,(b\cdot\nabla_y\varphi)\,\mathrm{d}y\mathrm{d}t=0.$$

Therefore $u(t,\cdot)\in\ker\mathcal{T}=\ker(b\cdot\nabla_y)$, $t\in\mathbb{R}_+$, cf. Proposition 3.3. Clearly (50) holds true for any $\varphi\in V$. In particular, for any $\varphi\in V\cap\ker(b\cdot\nabla_y)$ one gets

$$-\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}\eta' u^{\varepsilon_k}\varphi\,\mathrm{d}y\mathrm{d}t-\eta(0)\int_{\mathbb{R}^m}u^{\varepsilon_k}_{\mathrm{in}}\varphi\,\mathrm{d}y+\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}\eta\nabla_y u^{\varepsilon_k}\cdot D\nabla_y\varphi\,\mathrm{d}y\mathrm{d}t=0.\tag{51}$$

Thanks to the average properties we have

$$\int_{\mathbb{R}^m}u^{\varepsilon_k}_{\mathrm{in}}\varphi\,\mathrm{d}y=\int_{\mathbb{R}^m}\left\langle u^{\varepsilon_k}_{\mathrm{in}}\right\rangle\varphi\,\mathrm{d}y\to\int_{\mathbb{R}^m}u_{\mathrm{in}}\varphi\,\mathrm{d}y$$

and thus, letting $k\to+\infty$ in (51), leads to

$$-\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}\eta' u\varphi\,\mathrm{d}y\mathrm{d}t-\eta(0)\int_{\mathbb{R}^m}u_{\mathrm{in}}\varphi\,\mathrm{d}y+\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}\eta\nabla_y u\cdot D\nabla_y\varphi\,\mathrm{d}y\mathrm{d}t=0.\tag{52}$$

Since $u(t,\cdot),\varphi\in V\cap\ker(b\cdot\nabla_y)$ we have cf. Theorem 2.1

$$\int_{\mathbb{R}^m}\nabla_y u\cdot D\nabla_y\varphi\,\mathrm{d}y=\int_{\mathbb{R}^m}\nabla_y u\cdot\langle D\rangle_Q\nabla_y\varphi\,\mathrm{d}y$$

and (52) becomes

$$-\int_0^{+\infty}\!\!\int_{\mathbb{R}^m} \eta' u\varphi \,\mathrm{d}y\mathrm{d}t - \eta(0)\int_{\mathbb{R}^m} u_{\mathrm{in}}\varphi \,\mathrm{d}y + \int_0^{+\infty}\!\!\int_{\mathbb{R}^m} \eta\nabla_y u\cdot \langle D\rangle_Q \nabla_y\varphi \,\mathrm{d}y\mathrm{d}t = 0. \tag{53}$$

But (53) is still valid for test functions $\varphi = b\cdot\nabla_y\psi$, $\psi\in C^2_c(\mathbb{R}^m)$ since $u(t,\cdot)\in\ker(b\cdot\nabla_y)$, $u_{\mathrm{in}} = \mathrm{w}-\lim_{\varepsilon\searrow 0}\langle u^\varepsilon_{\mathrm{in}}\rangle \in \ker(b\cdot\nabla_y)$ and $\langle D\rangle_Q\in\ker L$

$$\int_{\mathbb{R}^m} u(t,y) b\cdot\nabla_y\psi \,\mathrm{d}y = 0,\;\; \int_{\mathbb{R}^m} u_{\mathrm{in}} b\cdot\nabla_y\psi\,\mathrm{d}y = 0,\;\; \int_{\mathbb{R}^m}\nabla_y u\cdot\langle D\rangle_Q\nabla_y(b\cdot\nabla_y\psi)\,\mathrm{d}y = 0$$

cf. Theorem 2.1. Therefore, for any $v\in V$ one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} u(t,y)v(y)\,\mathrm{d}y + \int_{\mathbb{R}^m}\nabla_y u\cdot\langle D\rangle_Q\nabla_y v\,\mathrm{d}y = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}^m)$$

with $u(0)=u_{\mathrm{in}}$. By the uniqueness of the solution of (23), (24) we deduce that all the family $(u^\varepsilon)_\varepsilon$ converges weakly to u. $\square$

Remark 4.1 *Notice that* (23) *propagates the constraint $b\cdot\nabla_y u = 0$, if satisfied initially. Indeed, for any $v\in C^1_c(\mathbb{R}^m)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} u(t,y)v(y)\,\mathrm{d}y + \int_{\mathbb{R}^m}\nabla_y u\cdot\langle D\rangle_Q\nabla_y v\,\mathrm{d}y = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}^m). \tag{54}$$

Since $\langle D\rangle_Q\in\ker L$, we know by the second statement of Proposition 3.9 that

$$\int_{\mathbb{R}^m}\nabla_y u_s\cdot\langle D\rangle_Q\nabla_y v\,\mathrm{d}y = \int_{\mathbb{R}^m}\nabla_y u\cdot\langle D\rangle_Q\nabla_y v_{-s}\,\mathrm{d}y.$$

Replacing v by v_{-s} in (54) *we obtain*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} u_s v\,\mathrm{d}y + \int_{\mathbb{R}^m}\nabla_y u_s\cdot\langle D\rangle_Q\nabla_y v\,\mathrm{d}y = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}^m)$$

and therefore u_s solves

$$\partial_t u_s - \mathrm{div}_y(\langle D\rangle_Q\nabla_y u_s) = 0,\;\; (t,y)\in\mathbb{R}_+\times\mathbb{R}^m$$

and $u_s(0,y) = u_{\mathrm{in}}(Y(s;y)) = u_{\mathrm{in}}(y), y\in\mathbb{R}^m$. By the uniqueness of the solution of (23), (24) *one gets $u_s = u$ and thus, at any time $t\in\mathbb{R}_+$, $b\cdot\nabla_y u(t,\cdot) = 0$.*

5 Second order approximation

For the moment we have determined the model satisfied by the dominant term in the expansion (5). We focus now on second order approximation, that is, a model which takes into account the first order correction term εu^1. Up to now we have used the equations (6), (7). Finding a closure for $u + \varepsilon u^1$ will require one more equation

$$\partial_t u^1 - \mathrm{div}_y(D\nabla_y u^1) - \mathrm{div}_y(b \otimes b \nabla_y u^2) = 0, \ \ (t, y) \in \mathbb{R}_+ \times \mathbb{R}^m. \tag{55}$$

Let us see, at least formally, how to get a second order approximation for $(u^\varepsilon)_\varepsilon$, when ε becomes small. The first order approximation *i.e.*, the closure for u, has been obtained by averaging (7) and by taking into account that $u \in \ker(b \cdot \nabla_y)$

$$\partial_t u = \langle \mathrm{div}_y(D\nabla_y u)\rangle = \mathrm{div}_y(\langle D\rangle_Q \nabla_y u).$$

Thus u^1 satisfies

$$\mathrm{div}_y(\langle D\rangle_Q \nabla_y u) - \mathrm{div}_y(D\nabla_y u) - \mathrm{div}_y(b \otimes b\nabla_y u^1) = 0 \tag{56}$$

from which we expect to express u^1, up to a function in $\ker(b \cdot \nabla_y)$, in terms of u. In order to do that we need Theorem 2.3.

Proof. (of Theorem 2.3) We claim that Range L^2 = Range L and thus Range L^2 is closed as well. Clearly Range $L^2 \subset$ Range L. Consider now $Z = L(Y)$ for some $Y \in \mathrm{dom}(L)$. But $Y - \mathrm{Proj}_{\ker L} Y \in \ker L^\perp = (\ker L^\star)^\perp = \overline{\mathrm{Range}\ L} = \mathrm{Range}\ L$ and there is $X \in \mathrm{dom}(L)$ such that $Y - \mathrm{Proj}_{\ker L} Y = L(X)$. Finally $X \in \mathrm{dom}(L^2)$ and

$$Z = L(Y) = L(Y - \mathrm{Proj}_{\ker L} Y) = L(L(X)).$$

By construction we have $D - \langle D\rangle_Q \in (\ker L)^\perp = (\ker L^\star)^\perp = \overline{\mathrm{Range}\ L} = \mathrm{Range}\ L = \mathrm{Range}\ L^2$ and thus there is a unique $F \in \mathrm{dom}(L^2) \cap (\ker L)^\perp$ such that $D = \langle D\rangle_Q - L(L(F))$. As $F \in (\ker L)^\perp$, there is $C \in \mathrm{dom}(L)$ such that $F = L(C)$ implying that ${}^tF = {}^tL(C) = L({}^tC)$. Therefore ${}^tF \in \mathrm{dom}(L^2) \cap (\ker L)^\perp$ and satisfies the same equation as F

$$L(L({}^tF)) = {}^tL(L(F)) = \langle D\rangle_Q - D.$$

By the uniqueness we deduce that F is a field of symmetric matrices. By Proposition 3.13 we know that

$$-\mathrm{div}_y(L(F)\nabla_y) = [b \cdot \nabla_y, -\mathrm{div}_y(F\nabla_y)] \ \text{ in } \ \mathcal{D}'(\mathbb{R}^m)$$

i.e.,

$$\int_{\mathbb{R}^m} L(F)\nabla_y u \cdot \nabla_y v \ \mathrm{d}y = -\int_{\mathbb{R}^m} F\nabla_y u \cdot \nabla_y(b \cdot \nabla_y v) \ \mathrm{d}y - \int_{\mathbb{R}^m} F\nabla_y(b \cdot \nabla_y u) \cdot \nabla_y v \ \mathrm{d}y$$

for any $u, v \in C^2_c(\mathbb{R}^m)$. Similarly, $E := L(F)$ satisfies

$$-\mathrm{div}_y(L^2(F)\nabla_y) = -\mathrm{div}_y(L(E)\nabla_y) = [b \cdot \nabla_y, -\mathrm{div}_y(E\nabla_y)] \ \text{ in } \ \mathcal{D}'(\mathbb{R}^m)$$

and thus, for any $u, v \in C_c^3(\mathbb{R}^m)$ one gets

$$\begin{aligned}
&\int_{\mathbb{R}^m} (\langle D \rangle_Q - D)\nabla_y u \cdot \nabla_y v \ \mathrm{d}y = \int_{\mathbb{R}^m} L^2(F)\nabla_y u \cdot \nabla_y v \ \mathrm{d}y \\
&= -\int_{\mathbb{R}^m} L(F)\nabla_y u \cdot \nabla_y (b \cdot \nabla_y v) \ \mathrm{d}y - \int_{\mathbb{R}^m} L(F)\nabla_y (b \cdot \nabla_y u) \cdot \nabla_y v \ \mathrm{d}y \\
&= \int_{\mathbb{R}^m} F\nabla_y u \cdot \nabla_y (b \cdot \nabla_y (b \cdot \nabla_y v)) \ \mathrm{d}y + \int_{\mathbb{R}^m} F\nabla_y (b \cdot \nabla_y u) \cdot \nabla_y (b \cdot \nabla_y v) \ \mathrm{d}y \\
&+ \int_{\mathbb{R}^m} F\nabla_y (b \cdot \nabla_y u) \cdot \nabla_y (b \cdot \nabla_y v) \ \mathrm{d}y + \int_{\mathbb{R}^m} F\nabla_y (b \cdot \nabla_y (b \cdot \nabla_y u)) \cdot \nabla_y v \ \mathrm{d}y.
\end{aligned}$$

□

The matrix fields $F \in \mathrm{dom}(L^2)$ and $E = L(F) \in \mathrm{dom}(L)$ enter the second order approximation model, and therefore we need to compute them. For that notice that we have the following properties.

Proposition 5.1 *For any $u, v \in C^1(\mathbb{R}^m)$ which are constant along the flow of b we have in $\mathcal{D}'(\mathbb{R}^m)$*

$$D\nabla_y u \cdot \nabla_y v - \langle D \rangle_Q \nabla_y u \cdot \nabla_y v = -b \cdot \nabla_y (E\nabla_y u \cdot \nabla_y v) = -\mathrm{div}_y (b \otimes b \nabla_y (F\nabla_y u \cdot \nabla_y v))$$

and

$$\langle E\nabla_y u \cdot \nabla_y v \rangle = \langle F\nabla_y u \cdot \nabla_y v \rangle = 0.$$

In particular

$$\int_{\mathbb{R}^m} E\nabla_y u \cdot \nabla_y v \ \mathrm{d}y = \int_{\mathbb{R}^m} \langle E\nabla_y u \cdot \nabla_y v \rangle \ \mathrm{d}y = 0$$

$$\int_{\mathbb{R}^m} F\nabla_y u \cdot \nabla_y v \ \mathrm{d}y = \int_{\mathbb{R}^m} \langle F\nabla_y u \cdot \nabla_y v \rangle \ \mathrm{d}y = 0$$

saying that $\langle \mathrm{div}_y (E\nabla_y u) \rangle = \langle \mathrm{div}_y (F\nabla_y u) \rangle = 0$ in $\mathcal{D}'(\mathbb{R}^m)$.

Proof. Consider $\varphi \in C_c^1(\mathbb{R}^m)$, $u, v \in C^1(\mathbb{R}^m)$ such that $u_s = u, v_s = v$, $s \in \mathbb{R}$ and the matrix field $U = \varphi Q^{-1}\nabla_y v \otimes \nabla_y u Q^{-1} \in H_Q$. Actually $U \in \mathrm{dom}(L)$ and, as in the proof of the last statement in Proposition 3.13, one gets

$$\begin{aligned}
L(U) &= (b \cdot \nabla_y \varphi) Q^{-1}\nabla_y v \otimes \nabla_y u Q^{-1} + \varphi \, L(Q^{-1}\nabla_y v \otimes \nabla_y u Q^{-1}) \\
&= (b \cdot \nabla_y \varphi) Q^{-1}\nabla_y v \otimes \nabla_y u Q^{-1}
\end{aligned}$$

since $Q^{-1}\nabla_y v \otimes \nabla_y u Q^{-1} \in \ker(L)$. Multiplying by U the equality $D - \langle D \rangle_Q = -L(E)$, $E = L(F)$, one gets

$$\int_{\mathbb{R}^m} \varphi (D - \langle D \rangle_Q)\nabla_y u \cdot \nabla_y v \ \mathrm{d}y = -(L(E), U)_Q = (E, L(U))_Q = \int_{\mathbb{R}^m} (b \cdot \nabla_y \varphi)(E\nabla_y u \cdot \nabla_y v) \ \mathrm{d}y$$

implying that $D\nabla_y u \cdot \nabla_y v = \langle D\rangle_Q \nabla_y u \cdot \nabla_y v - b \cdot \nabla_y (E\nabla_y u \cdot \nabla_y v)$ in $\mathcal{D}'(\mathbb{R}^m)$. Multiplying by U the equality $E = L(F)$ yields

$$\int_{\mathbb{R}^m} \varphi E\nabla_y u\cdot\nabla_y v \,\mathrm{d}y = (E,U)_Q = (L(F),U)_Q = -(F, L(U))_Q = -\int_{\mathbb{R}^m} (b\cdot\nabla_y\varphi) F\nabla_y u\cdot\nabla_y v \,\mathrm{d}y.$$

We obtain

$$E\nabla_y u \cdot \nabla_y v = b \cdot \nabla_y (F\nabla_y u \cdot \nabla_y v) \ \text{ in } \ \mathcal{D}'(\mathbb{R}^m)$$

and thus

$$D\nabla_y u \cdot \nabla_y v - \langle D\rangle_Q \nabla_y u \cdot \nabla_y v = -b \cdot \nabla_y (E\nabla_y u \cdot \nabla_y v) = -b \cdot \nabla_y (b \cdot \nabla_y (F\nabla_y u \cdot \nabla_y v))$$

in $\mathcal{D}'(\mathbb{R}^m)$. Consider now $U = \varphi Q^{-1}\nabla_y v \otimes \nabla_y u Q^{-1}$ with $\varphi \in \ker(b \cdot \nabla_y)$. We know that $L(U) = 0$ and since, by construction $F \in (\ker L)^\perp$, we deduce

$$\int_{\mathbb{R}^m} \varphi F\nabla_y u \cdot \nabla_y v \,\mathrm{d}y = (F,U)_Q = 0$$

saying that $\langle F\nabla_y u \cdot \nabla_y v\rangle = 0$. Similarly $E = L(F) \in (\ker L)^\perp$ and $\langle E\nabla_y u \cdot \nabla_y v\rangle = 0$. $\square$

The matrix fields E and F, constructed in Theorem 2.3, can be computed easily, point by point, under the same hypotheses which guarantee the explicit computation of the averaged matrix field, see next remark.

Remark 5.1 *Assume that there is u_0 (eventually multi-valued) satisfying $u_0(Y(s;y)) = u_0(y) + s$, $s \in \mathbb{R}, y \in \mathbb{R}^m$. Its gradient changes along the flow of b exactly as the gradient of any function which is constant along this flow cf. Remark 3.7. We deduce that $Q^{-1}\nabla_y v \otimes \nabla_y u Q^{-1} \in \ker L$ for any $u, v \in \ker(b \cdot \nabla_y) \cup \{u_0\}$ and therefore the arguments in the proof of Proposition 5.1 still apply when $u, v \in \ker(b \cdot \nabla_y) \cup \{u_0\}$. In the case when $m-1$ independent prime integrals $\{u_1, ..., u_{m-1}\}$ of b are known, the matrix fields E, F come, by imposing for any $i, j \in \{0, 1, ..., m-1\}$*

$$-b \cdot \nabla_y (E\nabla_y u_i \cdot \nabla_y u_j) = D\nabla_y u_i \cdot \nabla_y u_j - \langle D\nabla_y u_i \cdot \nabla_y u_j\rangle, \ \ \langle E\nabla_y u_i \cdot \nabla_y u_j\rangle = 0$$

and

$$b \cdot \nabla_y (F\nabla_y u_i \cdot \nabla_y u_j) = E\nabla_y u_i \cdot \nabla_y u_j, \ \ \langle F\nabla_y u_i \cdot \nabla_y u_j\rangle = 0.$$

We indicate now sufficient conditions which guarantee that the range of L is closed. Basically we will prove that, up to a isomorphism, the operator L on H_Q reduces to the operator $b\cdot\nabla_y$ on $L^2(\mathbb{R}^m)$. In that case if the range of $b \cdot \nabla_y$ is closed, then so is the range of L.

Proposition 5.2 *Assume that* (14), (15), (29) *hold true and that there is a matrix field $R(y)$ such that* (25) *holds true. Then the range of L is closed.*

Proof. Observe that (25) implies (18). Indeed, it is easily seen that $b \cdot \nabla_y R + R\partial_y b = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ is equivalent to $R = R_s \partial_y Y(s;\cdot)$, $s \in \mathbb{R}$. We deduce that $P = R^{-1}\,{}^t R^{-1}$ satisfies

$$G(s)P = \partial_y Y^{-1}(s;\cdot) P_s {}^t\partial_y Y^{-1}(s;\cdot) = \partial_y Y^{-1}(s;\cdot) R_s^{-1}\,{}^t R_s{}^{-1}\,{}^t\partial_y Y^{-1}(s;\cdot) = R^{-1}\,{}^t R^{-1} = P$$

saying that $[b, P] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. Therefore we can define L as before, on H_Q, which coincides in this case with $\{A : RA\,{}^t R \in L^2(\mathbb{R}^m)\}$. We claim that $i \circ L = (b \cdot \nabla_y) \circ i$ where $i : H_Q \to L^2(\mathbb{R}^m)$, $i(A) = RA\,{}^t R$, $A \in H_Q$, which comes immediately from the equalities

$$(i \circ G(s))A = RG(s)A^t R = R\partial_y Y^{-1}(s;\cdot) A_s {}^t\partial_y Y^{-1t} R = R_s A_s {}^t R_s = (i(A))_s,\ s \in \mathbb{R}, A \in H_Q.$$

In particular we have

$$\ker L = \{A \in H_Q\ :\ i(A) \in \ker(b \cdot \nabla_y)\}$$

and

$$\begin{aligned}(\ker L)^\perp &= \{A \in H_Q\ :\ \int_{\mathbb{R}^m} i(A) : U \,\mathrm{d}y = 0\ \forall\, U \in \ker(b \cdot \nabla_y)\}\\ &= \{A \in H_Q\ :\ i(A) \in (\ker(b \cdot \nabla_y))^\perp\}.\end{aligned}$$

For any $A \in (\ker L)^\perp$ we can apply the Poincaré inequality (29) to $i(A) \in (\ker(b \cdot \nabla_y))^\perp$ and we obtain

$$|A|_Q = |i(A)|_{L^2} \le C_P |b \cdot \nabla_y(i(A))|_{L^2} = C_P |i(L(A))|_{L^2} = C_P |L(A)|_Q.$$

Therefore L satisfies a Poincaré inequality as well, and thus the range of L is closed. $\square$

Remark 5.2 *The hypothesis $b \cdot \nabla_y R + R\partial_y b = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ says that the columns of R^{-1} form a family of m independent vector fields in involution with respect to b, cf. Proposition 3.4*

$$R_s^{-1}(y) = \partial_y Y(s;y) R^{-1}(y),\ \ s \in \mathbb{R},\ \ y \in \mathbb{R}^m.$$

Remark 5.3 *For any $U \in \ker L$, that is $i(U) \in \ker(b \cdot \nabla_y)$, we have*

$$\int_{\mathbb{R}^m} R(D - \langle D \rangle_Q)^t R : i(U)\,\mathrm{d}y = 0.$$

As $\langle D \rangle_Q \in \ker L$, we know that $i(\langle D \rangle_Q) = R\,\langle D \rangle_Q {}^t R \in \ker(b \cdot \nabla_y)$ and thus the matrix field $R\,\langle D \rangle_Q {}^t R$ is the average (along the flow of b) of the matrix field $RD\,{}^t R$, which allows us to express $\langle D \rangle_Q$ in terms of R and D

$$R\,\langle D \rangle_Q {}^t R = \left\langle RD\,{}^t R \right\rangle.$$

From now on we assume that (25) holds true. Applying the decomposition of Theorem 2.3 with the dominant term $u \in \ker(b \cdot \nabla_y)$ in the expansion (5) and any $v \in C_c^3(\mathbb{R}^m)$ yields

$$\int_{\mathbb{R}^m} (D - \langle D \rangle_Q) \nabla_y u \cdot \nabla_y v \, \mathrm{d}y = - \int_{\mathbb{R}^m} F \nabla_y u \cdot \nabla_y (b \cdot \nabla_y (b \cdot \nabla_y v)) \, \mathrm{d}y.$$

From (56) one gets

$$\int_{\mathbb{R}^m} (D - \langle D \rangle_Q) \nabla_y u \cdot \nabla_y v \, \mathrm{d}y - \int_{\mathbb{R}^m} u^1 b \cdot \nabla_y (b \cdot \nabla_y v) \, \mathrm{d}y = 0$$

and thus

$$u^1 = \mathrm{div}_y(F \nabla_y u) + v^1, \quad v^1 \in \ker(b \cdot \nabla_y (b \cdot \nabla_y)) = \ker(b \cdot \nabla_y). \tag{57}$$

Notice that $\langle u^1 \rangle = v^1$, since $\langle \mathrm{div}_y(F \nabla_y u) \rangle = 0$, cf. Proposition 5.1. The time evolution for $v^1 = \langle u^1 \rangle$ comes by averaging (55)

$$\partial_t v^1 - \langle \mathrm{div}_y(D \nabla_y v^1) \rangle - \langle \mathrm{div}_y(D \nabla_y (\mathrm{div}_y(F \nabla_y u))) \rangle = 0.$$

As $v^1 \in \ker(b \cdot \nabla_y)$ we have

$$- \langle \mathrm{div}_y(D \nabla_y v^1) \rangle = - \mathrm{div}_y(\langle D \rangle_Q \nabla_y v^1)$$

and we can write, with the notation $w^1 = \mathrm{div}_y(F \nabla_y u)$

$$\partial_t \{u + \varepsilon u^1\} - \mathrm{div}_y(\langle D \rangle_Q \nabla_y \{u + \varepsilon u^1\}) = \varepsilon \partial_t w^1 - \varepsilon \mathrm{div}_y(\langle D \rangle_Q \nabla_y w^1) + \varepsilon \langle \mathrm{div}_y(D \nabla_y w^1) \rangle. \tag{58}$$

But the time derivative of w^1 is given by

$$\partial_t w^1 = \mathrm{div}_y(F \nabla_y \partial_t u) = \mathrm{div}_y(F \nabla_y (\mathrm{div}_y(\langle D \rangle_Q \nabla_y u)))$$

which implies

$$\begin{aligned} \partial_t w^1 - \mathrm{div}_y(\langle D \rangle_Q \nabla_y w^1) &= \mathrm{div}_y(F \nabla_y (\mathrm{div}_y(\langle D \rangle_Q \nabla_y u))) - \mathrm{div}_y(\langle D \rangle_Q \nabla_y (\mathrm{div}_y(F \nabla_y u))) \\ &= -[\mathrm{div}_y(\langle D \rangle_Q \nabla_y), \mathrm{div}_y(F \nabla_y)] u. \end{aligned}$$

Up to a second order term, the equation (58) writes

$$\begin{aligned} \partial_t \{u + \varepsilon u^1\} - \mathrm{div}_y(\langle D \rangle_Q \nabla_y \{u + \varepsilon u^1\}) + \varepsilon [\mathrm{div}_y(\langle D \rangle_Q \nabla_y), \mathrm{div}_y(F \nabla_y)] \{u + \varepsilon u^1\} \\ - \varepsilon \langle \mathrm{div}_y(D \nabla_y (\mathrm{div}_y(F \nabla_y u))) \rangle = \mathcal{O}(\varepsilon^2). \end{aligned} \tag{59}$$

We claim that for any $u \in \ker(b \cdot \nabla_y)$ we have

$$\langle \mathrm{div}_y(D \nabla_y (\mathrm{div}_y(F \nabla_y u))) \rangle = \langle \mathrm{div}_y(E \nabla_y (\mathrm{div}_y(E \nabla_y u))) \rangle. \tag{60}$$

By Proposition 5.1 we know that $\langle \mathrm{div}_y(F\nabla_y u)\rangle = 0$. As $L(\langle D\rangle_Q) = 0$ we have

$$[b\cdot\nabla_y, -\mathrm{div}_y(\langle D\rangle_Q \nabla_y)] = -\mathrm{div}_y(L(\langle D\rangle_Q)\nabla_y) = 0$$

and thus $\mathrm{div}_y(\langle D\rangle_Q \nabla_y)$ leaves invariant the subspace of functions which are constant along the flow of b. By the symmetry of the operator $\mathrm{div}_y(\langle D\rangle_Q \nabla_y)$, we deduce that the subspace of zero average functions is also left invariant by $\mathrm{div}_y(\langle D\rangle_Q \nabla_y)$. Therefore we have $\left\langle \mathrm{div}_y(\langle D\rangle_Q \nabla_y(\mathrm{div}_y(F\nabla_y u)))\right\rangle = 0$ and

$$\langle \mathrm{div}_y(D\nabla_y(\mathrm{div}_y(F\nabla_y u)))\rangle = \left\langle \mathrm{div}_y((D - \langle D\rangle_Q)\nabla_y(\mathrm{div}_y(F\nabla_y u)))\right\rangle.$$

Thanks to Theorem 2.3 we have

$$\begin{aligned}
\mathrm{div}_y((D - \langle D\rangle_Q)\nabla_y) &= [b\cdot\nabla_y, [b\cdot\nabla_y, -\mathrm{div}_y(F\nabla_y)]\,] \\
&= [b\cdot\nabla_y, -\mathrm{div}_y(L(F)\nabla_y)] \\
&= [b\cdot\nabla_y, -\mathrm{div}_y(E\nabla_y)]
\end{aligned}$$

which implies that

$$\begin{aligned}
&\langle \mathrm{div}_y(D\nabla_y(\mathrm{div}_y(F\nabla_y u)))\rangle = \left\langle \mathrm{div}_y((D - \langle D\rangle_Q)\nabla_y(\mathrm{div}_y(F\nabla_y u)))\right\rangle \\
&= \langle \mathrm{div}_y(E\nabla_y(b\cdot\nabla_y(\mathrm{div}_y(F\nabla_y u)))) - b\cdot\nabla_y(\mathrm{div}_y(E\nabla_y(\mathrm{div}_y(F\nabla_y u))))\rangle \\
&= \langle \mathrm{div}_y(E\nabla_y(b\cdot\nabla_y(\mathrm{div}_y(F\nabla_y u))))\rangle\,.
\end{aligned}$$

Finally notice that

$$-\mathrm{div}_y(E\nabla_y u) = -\mathrm{div}_y(L(F)\nabla_y u) = [b\cdot\nabla_y, -\mathrm{div}_y(F\nabla_y u)] = -b\cdot\nabla_y(\mathrm{div}_y(F\nabla_y u))$$

and (60) follows. We need to average the differential operator $\mathrm{div}_y(E\nabla_y(\mathrm{div}_y(E\nabla_y)))$ on functions $u \in \ker(b\cdot\nabla_y)$. This is not a trivial task, due to the high order of derivatives entering this operator (nevertheless, any other differential operator can be treated in a similar way). For simplicity we perform these computations at a formal level, assuming that all fields are smooth enough. The idea is to express the above differential operator in terms of the derivations ${}^tR^{-1}\nabla_y$ which commute with the average operator (see Proposition 3.6), since the columns of R^{-1} contain vector fields in involution with $b(y)$.

Lemma 5.1 *Under the hypothesis* (25), *for any smooth function* $u(y)$ *and matrix field* $E(y)$ *we have*

$$\mathrm{div}_y(E\nabla_y u) = \mathrm{div}_y(R\,{}^tE)\cdot({}^tR^{-1}\nabla_y u) + RE\,{}^tR : ({}^tR^{-1}\nabla_y \otimes {}^tR^{-1}\nabla_y)u. \quad (61)$$

Proof. Applying the formula $\operatorname{div}_y(A\xi) = \operatorname{div}_y {}^t A \cdot \xi + {}^t A : \partial_y \xi$, where $A(y)$ is a matrix field and $\xi(y)$ is a vector field, one gets

$$\operatorname{div}_y(E\nabla_y u) = \operatorname{div}_y(E\,{}^tR\,{}^tR^{-1}\nabla_y u) = \operatorname{div}_y(R\,{}^tE) \cdot ({}^tR^{-1}\nabla_y u) + R\,{}^tE : \partial_y({}^tR^{-1}\nabla_y u).$$

The last term in the above formula writes

$$\begin{aligned} R\,{}^tE : \partial_y({}^tR^{-1}\nabla_y u) &= R\,{}^tE\,{}^tR\,{}^tR^{-1} : \partial_y({}^tR^{-1}\nabla_y u) \\ &= R\,{}^tE\,{}^tR : \partial_y({}^tR^{-1}\nabla_y u)R^{-1} \\ &= RE\,{}^tR : {}^tR^{-1}\,{}^t\partial_y({}^tR^{-1}\nabla_y u) \\ &= RE\,{}^tR : ({}^tR^{-1}\nabla_y \otimes {}^tR^{-1}\nabla_y)u \end{aligned}$$

and (61) follows. $\square$

Next we claim that the term $\langle \operatorname{div}_y(E\nabla_y(\operatorname{div}_y(E\nabla_y u)))\rangle$ reduces to a differential operator, if $u \in \ker(b \cdot \nabla_y)$ (see Appendix A for proof details).

Proposition 5.3 *Under the hypothesis* (25), *for any smooth matrix field E there is a linear differential operator $S(u)$ of order four, such that, for any smooth $u \in \ker(b \cdot \nabla_y)$*

$$\langle \operatorname{div}_y(E\nabla_y(\operatorname{div}_y(E\nabla_y u)))\rangle = S(u). \tag{62}$$

Combining (59), (60), (62) we obtain

$$\begin{aligned} \partial_t\{u + \varepsilon u^1\} - \operatorname{div}_y(\langle D\rangle_Q \nabla_y\{u + \varepsilon u^1\}) + \varepsilon[\operatorname{div}_y(\langle D\rangle_Q \nabla_y), \operatorname{div}_y(F\nabla_y)]\{u + \varepsilon u^1\} \\ - \varepsilon S(u + \varepsilon u^1) = \mathcal{O}(\varepsilon^2) \end{aligned}$$

which justifies the equation introduced in (26). The initial condition comes formally by averaging the Ansatz (5)

$$\langle u^\varepsilon \rangle = u + \varepsilon v^1 + \mathcal{O}(\varepsilon^2).$$

One gets

$$v^1(0,\cdot) = \text{w-}\lim_{\varepsilon \searrow 0} \frac{\langle u^\varepsilon_{\text{in}} \rangle - u_{\text{in}}}{\varepsilon} = v_{\text{in}}$$

implying that $u^1(0,\cdot) = v_{\text{in}} + \operatorname{div}_y(F\nabla_y u_{\text{in}})$, cf. (57), which justifies (27).

6 Examples

Let us consider the vector field $b(y) = {}^\perp y := (y_2, -y_1)$, for any $y = (y_1, y_2) \in \mathbb{R}^2$ and the matrix field

$$D(y) = \begin{pmatrix} \lambda_1(y) & 0 \\ 0 & \lambda_2(y) \end{pmatrix}, \quad y \in \mathbb{R}^2$$

where λ_1, λ_2 are given functions, satisfying $\min_{y\in\mathbb{R}^2}\{\lambda_1(y), \lambda_2(y)\} \geq d > 0$. We intend to determine the first order approximation, when $\varepsilon \searrow 0$, for the heat equation

$$\partial_t u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) - \frac{1}{\varepsilon}\mathrm{div}_y(b(y)\otimes b(y)\nabla_y u^\varepsilon) = 0, \ \ (t,y) \in \mathbb{R}_+ \times \mathbb{R}^2 \tag{63}$$

with the initial condition

$$u^\varepsilon(0,y) = u_{\mathrm{in}}(y), \ \ y \in \mathbb{R}^2.$$

The flow of b is given by $Y(s;y) = \mathcal{R}(-s)y$, $s \in \mathbb{R}, y \in \mathbb{R}^2$ where $\mathcal{R}(\alpha)$ stands for the rotation of angle $\alpha \in \mathbb{R}$. The functions in $\ker(b\cdot\nabla_y)$ are those depending only on $|y|$. Notice that the matrix field

$$R(y) = \frac{1}{|y|}\begin{pmatrix} y_2 & -y_1 \\ y_1 & y_2 \end{pmatrix}$$

satisfies $b \cdot \nabla_y R + R\partial_y b = 0$ and $Q = {}^tRR = I_2$. The averaged matrix field $\langle D\rangle_Q$ comes, thanks to Remark 5.3, by the formula $R\langle D\rangle_Q {}^tR = \langle RD\,{}^tR\rangle$ and thus

$$\langle D\rangle_Q = {}^tR\langle RD\,{}^tR\rangle R, \ \ \langle RD\,{}^tR\rangle = \begin{pmatrix} \left\langle\frac{\lambda_1 y_2^2+\lambda_2 y_1^2}{|y|^2}\right\rangle & \left\langle\frac{(\lambda_1-\lambda_2)y_1y_2}{|y|^2}\right\rangle \\ \left\langle\frac{(\lambda_1-\lambda_2)y_1y_2}{|y|^2}\right\rangle & \left\langle\frac{\lambda_1 y_1^2+\lambda_2 y_2^2}{|y|^2}\right\rangle \end{pmatrix}.$$

In the case when λ_1, λ_2 are left invariant by the flow of b, that is λ_1, λ_2 depend only on $|y|$, it is easily seen that

$$\left\langle\frac{y_1^2}{|y|^2}\right\rangle = \left\langle\frac{y_2^2}{|y|^2}\right\rangle = \frac{1}{2}, \ \ \left\langle\frac{y_1y_2}{|y|^2}\right\rangle = 0$$

and thus

$$\langle D\rangle_Q = {}^tR\frac{\lambda_1+\lambda_2}{2}I_2R = \frac{\lambda_1+\lambda_2}{2}I_2.$$

The first order approximation of (63) is given by

$$\begin{cases} \partial_t u - \mathrm{div}_y\left(\frac{\lambda_1(y)+\lambda_2(y)}{2}\nabla_y u\right) = 0, & (t,y) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ u(0,y) = u_{\mathrm{in}}(y), & y \in \mathbb{R}^2. \end{cases}$$

We consider the multi-valued function $u_0(y) = -\theta(y)$, where $y = |y|(\cos\theta(y), \sin\theta(y))$, which satisfies $b\cdot\nabla_y u_0 = 1$, or $u_0(Y(s;y)) = u_0(y) + s$. Notice that the averaged matrix field $\langle D\rangle_Q$ satisfies (with $u_1(y) = |y|^2/2 \in \ker(b\cdot\nabla_y)$)

$$\nabla_y u_i \cdot \langle D\rangle_Q \nabla_y u_j = \langle\nabla_y u_i \cdot D\nabla_y u_j\rangle, \ \ i,j \in \{0,1\}$$

as predicted by Remark 3.7. In order to write the second order approximation, we need to compute the matrix fields E and F. By Remark 5.1 we have for any $i,j \in \{0,1\}$

$$-b\cdot\nabla_y(E\nabla_y u_i\cdot\nabla_y u_j) = D\nabla_y u_i\cdot\nabla_y u_j - \langle D\nabla_y u_i\cdot\nabla_y u_j\rangle, \ \ \langle E\nabla_y u_i\cdot\nabla_y u_j\rangle = 0$$

$$b \cdot \nabla_y (F \nabla_y u_i \cdot \nabla_y u_j) = E \nabla_y u_i \cdot \nabla_y u_j, \ \ \langle F \nabla_y u_i \cdot \nabla_y u_j \rangle = 0$$

leading to

$$E(y) = \frac{\lambda_1 - \lambda_2}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \ F(y) = \frac{\lambda_1 - \lambda_2}{8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For simplicity, for now on we assume that λ_1, λ_2 are constant, saying that $\mathrm{div}_y(\langle D \rangle_Q \nabla_y)$ and $\mathrm{div}_y(F \nabla_y)$ are differential operators with constant coefficients. Therefore their commutator vanishes

$$[\mathrm{div}_y(\langle D \rangle_Q \nabla_y), \mathrm{div}_y(F \nabla_y)] = 0.$$

It remains to compute the average of $\mathrm{div}_y \{E \nabla_y [\mathrm{div}_y (E \nabla_y)]\}$ on functions $u \in \ker(b \cdot \nabla_y)$. A direct computation shows that for any function $u \in \ker(b \cdot \nabla_y)$ (that is, for any function $u(y) = U(|y|^2/2)$) we have

$$\langle \mathrm{div}_y \{E \nabla_y [\mathrm{div}_y (E \nabla_y u)]\} \rangle = \frac{(\lambda_1 - \lambda_2)^2}{32} \Delta_y (\Delta_y u).$$

In this case, by Theorem 2.4, we obtain the second order model

$$\partial_t \tilde{u}^\varepsilon - \frac{\lambda_1 + \lambda_2}{2} \Delta_y \tilde{u}^\varepsilon - \varepsilon \frac{(\lambda_1 - \lambda_2)^2}{32} \Delta_y (\Delta_y \tilde{u}^\varepsilon) = 0.$$

We consider now the problem related to the anisotropic diffusion of the temperature inside a tokamak. In the two dimensional case, a divergence free magnetic field writes $b(y) = {}^\perp \nabla_y u_1, y \in \mathbb{R}^2$ for some function u_1 (the previous case corresponds to the particular function $u_1(y) = |y|^2/2, y \in \mathbb{R}^2$). In the general case we detail only the first order approximation. We suppose that there is a (multi-valued) function $u_0(y)$ such that $b \cdot \nabla_y u_0 = 1$ (indeed, u_0 can not be smooth everywhere on $\mathbb{R}^2$, otherwise $1 = \langle b \cdot \nabla_y u_0 \rangle = 0$). Since the parallel diffusion along the magnetic lines is much larger than the perpendicular diffusion, we are led to the equation

$$\partial_t u^\varepsilon - \mathrm{div}_y (D(y) \nabla_y u^\varepsilon) - \frac{1}{\varepsilon} \mathrm{div}_y (b(y) \otimes b(y) \nabla_y u^\varepsilon) = 0, \ \ (t, y) \in \mathbb{R}_+ \times \mathbb{R}^2$$

with

$$D(y) = I_2 - \frac{b(y) \otimes b(y)}{|b(y)|^2} = \frac{{}^\perp b(y) \otimes {}^\perp b(y)}{|b(y)|^2}.$$

We need to compute $\langle D \rangle_Q$. For that, we introduce the vector field

$$b_1 = (I_2 - b \otimes \nabla_y u_0) \frac{\nabla_y u_1}{|\nabla_y u_1|^2}$$

that is, the vector field satisfying $b_1 \cdot \nabla_y u_0 = 0, b_1 \cdot \nabla_y u_1 = 1$. Notice that, with the notation $b_0 := b$, we have

$$b_i \cdot \nabla_y u_j = \delta_{ij}, \ \ i, j \in \{0, 1\}$$

saying that at any point $y \in \mathbb{R}^2$, $\{b_0(y), b_1(y)\}$ is the dual basis of $\{\nabla_y u_0(y), \nabla_y u_1(y)\}$. It is easily seen that for any matrix field $M(y)$ we have the decomposition

$$M(y) = \sum_{0 \le i,j \le 1} (M(y) : \nabla_y u_i \otimes \nabla_y u_j)\, b_i \otimes b_j.$$

In particular, thanks to Remark 3.7

$$\langle D \rangle_Q = \sum_{0 \le i,j \le 1} (\langle D \rangle_Q : \nabla_y u_i \otimes \nabla_y u_j)\, b_i \otimes b_j = \sum_{0 \le i,j \le 1} \langle D : \nabla_y u_i \otimes \nabla_y u_j \rangle \; b_i \otimes b_j$$

and thus we need to compute the coefficients $\langle D : \nabla_y u_i \otimes \nabla_y u_j \rangle$, $i, j \in \{0, 1\}$. We obtain

$$\langle D : \nabla_y u_0 \otimes \nabla_y u_0 \rangle = \left\langle \frac{(\nabla_y u_0 \cdot \nabla_y u_1)^2}{|\nabla_y u_1|^2} \right\rangle, \quad \langle D : \nabla_y u_1 \otimes \nabla_y u_1 \rangle = \left\langle |\nabla_y u_1|^2 \right\rangle$$

$$\langle D : \nabla_y u_0 \otimes \nabla_y u_1 \rangle = \langle D : \nabla_y u_1 \otimes \nabla_y u_0 \rangle = \langle \nabla_y u_0 \cdot \nabla_y u_1 \rangle$$

implying that

$$\langle D \rangle_Q = \left\langle \frac{(\nabla_y u_0 \cdot \nabla_y u_1)^2}{|\nabla_y u_1|^2} \right\rangle b_0 \otimes b_0 + \langle \nabla_y u_0 \cdot \nabla_y u_1 \rangle (b_0 \otimes b_1 + b_1 \otimes b_0) + \left\langle |\nabla_y u_1|^2 \right\rangle \; b_1 \otimes b_1$$

and the first order approximation becomes $\partial_t u - \mathrm{div}_y(\langle D \rangle_Q \nabla_y u) = 0$.

A Proofs of Propositions 3.4, 3.5, 3.8, 3.9, 3.14, 5.3

Proof. (of Proposition 3.4) For simplicity we assume that b is divergence free. The general case follows similarly. Let $c(y)$ be a vector field satisfying (30). For any vector field $\phi \in C^1_c(\mathbb{R}^m)$ we have, with the notation $u_\tau = u(Y(\tau; \cdot))$

$$\int_{\mathbb{R}^m} c \cdot (\phi_{-h} - \phi)\, \mathrm{d}y = \int_{\mathbb{R}^m} (c_h - c) \cdot \phi\, \mathrm{d}y = \int_{\mathbb{R}^m} (\partial_y Y(h; y) - I) c \cdot \phi\, \mathrm{d}y.$$

Multiplying by h^{-1} and passing to the limit when $h \to 0$ imply

$$- \int_{\mathbb{R}^m} c(b \cdot \nabla_y \phi)\, \mathrm{d}y = \int_{\mathbb{R}^m} \partial_y b c \cdot \phi\, \mathrm{d}y$$

and therefore $(b \cdot \nabla_y) c - \partial_y b c = 0$ in $\mathcal{D}'(\mathbb{R}^m)$.

Conversely, assume that $[b, c] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. We introduce $e(s, y) = c(Y(s; y)) - \partial_y Y(s; y) c(y)$. Notice that $e(s, \cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^m), s \in \mathbb{R}$ and $e(0, \cdot) = 0$. For any vector field $\phi \in C^1_c(\mathbb{R}^m)$ we have

$$E_\phi(s) := \int_{\mathbb{R}^m} e(s, y) \cdot \phi(y)\, \mathrm{d}y = \int_{\mathbb{R}^m} c(y) \cdot \phi_{-s}\, \mathrm{d}y - \int_{\mathbb{R}^m} \partial_y Y(s; y) c(y) \cdot \phi(y)\, \mathrm{d}y$$

and thus

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}E_\phi(s) &= -\int_{\mathbb{R}^m} c(y)\cdot((b\cdot\nabla_y)\phi)_{-s}\,\mathrm{d}y - \int_{\mathbb{R}^m}\partial_y(b(Y(s;y)))\,c(y)\cdot\phi(y)\,\mathrm{d}y\\
&= -\int_{\mathbb{R}^m} c\cdot(b\cdot\nabla_y)\phi_{-s}\,\mathrm{d}y - \int_{\mathbb{R}^m}\partial_y b(Y(s;y))\partial_y Y(s;y)c(y)\cdot\phi(y)\,\mathrm{d}y\\
&= \int_{\mathbb{R}^m}\partial_y b\; c(y)\cdot\phi_{-s}\,\mathrm{d}y - \int_{\mathbb{R}^m}\partial_y b(Y(s;y))\partial_y Y(s;y)c(y)\cdot\phi(y)\,\mathrm{d}y\\
&= \int_{\mathbb{R}^m}\partial_y b(Y(s;y))(c(Y(s;y)) - \partial_y Y(s;y)c(y))\cdot\phi(y)\,\mathrm{d}y\\
&= \int_{\mathbb{R}^m} e(s,y)\cdot{}^t\partial_y b(Y(s;y))\phi(y)\,\mathrm{d}y.
\end{aligned}$$

In the previous computation we have used the fact that the derivation and tranlation along b commute

$$((b\cdot\nabla_y)\phi)_{-s} = (b\cdot\nabla_y)\phi_{-s}.$$

After integration with respect to s one gets

$$E_\phi(s) = \int_0^s\int_{\mathbb{R}^m} e(\tau,y)\cdot{}^t\partial_y b(Y(\tau;y))\phi(y)\,\mathrm{d}y\,\mathrm{d}\tau.$$

Clearly, the above equality still holds true for any $\phi\in C_c(\mathbb{R}^m)$. Consider $R>0, T>0$ and let $K = \|{}^t\partial_y b\circ Y\|_{L^\infty([-T,T]\times B_R)}$. Therefore, for any $s\in[-T,T]$ we obtain

$$\begin{aligned}
\|e(s,\cdot)\|_{L^\infty(B_R)} &= \sup\{|E_\phi(s)|\ :\ \phi\in C_c(B_R),\ \ \|\phi\|_{L^1(\mathbb{R}^m)}\le 1\}\\
&\le K\left|\int_0^s \|e(\tau,\cdot)\|_{L^\infty(B_R)}\mathrm{d}\tau\right|.
\end{aligned}$$

By Gronwall lemma we deduce that $\|e(s,\cdot)\|_{L^\infty(B_R)} = 0$ for $-T\le s\le T$ saying that $c(Y(s;y)) - \partial_y Y(s;y)c(y) = 0, s\in\mathbb{R}, y\in\mathbb{R}^m$. □

Proof. (of Proposition 3.5)

1. $\Longrightarrow$ 2. By Proposition 3.4 we deduce that $c(Y(s;y)) = \partial_y Y(s;y)c(y)$ and therefore

$$\begin{aligned}
\int_{\mathbb{R}^m}(c\cdot\nabla_y u)v_{-s}\,\mathrm{d}y &= \int_{\mathbb{R}^m} c(Y(s;y))\cdot(\nabla_y u)(Y(s;y))v(y)\,\mathrm{d}y\\
&= \int_{\mathbb{R}^m} c(y)\cdot{}^t\partial_y Y(s;y)(\nabla_y u)(Y(s;y))v(y)\,\mathrm{d}y = \int_{\mathbb{R}^m}(c(y)\cdot\nabla_y u_s)v(y)\,\mathrm{d}y.
\end{aligned}$$

2. $\Longrightarrow$ 3. Taking the derivative with respect to s of (31) at $s=0$, we obtain (32). 3. $\Longrightarrow$ 1. Applying (32) with $v\in C^1_c(\mathbb{R}^m)$ and $u_i = y_i\varphi(y)$, $\varphi\in C^2_c(\mathbb{R}^m)$, $\varphi=1$ on the support of v, yields

$$\int_{\mathbb{R}^m} c_i\; b\cdot\nabla_y v\,\mathrm{d}y + \int_{\mathbb{R}^m} c\cdot\nabla_y b_i\; v(y)\,\mathrm{d}y = 0$$

saying that $b\cdot\nabla_y c_i = (\partial_y b\; c)_i$ in $\mathcal{D}'(\mathbb{R}^m)$, $i\in\{1,...,m\}$ and thus $[b,c] = b\cdot\nabla_y c - \partial_y bc = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. □

Proof. (of Proposition 3.8) The arguments are very similar to those in the proof of Proposition 3.4. Let us give the main lines. We assume that b is divergence free, for simplicity. Let $A(y)$ be a matrix field satisfying (34). For any matrix field $U \in C_c^1(\mathbb{R}^m)$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^m} A(y) : (U(Y(-h;y)) - U(y)) \, \mathrm{d}y &= \int_{\mathbb{R}^m} (A(Y(h;y)) - A(y)) : U(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} (\partial_y Y(h;y) A(y) {}^t\partial_y Y(h;y) - A(y)) : U(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} \{(\partial_y Y(h;y) - I) A(y) {}^t\partial_y Y(h;y) : U(y) + A(y) {}^t(\partial_y Y(h;y) - I) : U(y)\} \, \mathrm{d}y.
\end{aligned}
$$

Multiplying by $\frac{1}{h}$ and passing $h \to 0$ we obtain

$$
- \int_{\mathbb{R}^m} A(y) : (b \cdot \nabla_y U) \, \mathrm{d}y = \int_{\mathbb{R}^m} (\partial_y b A(y) + A(y) {}^t\partial_y b) : U(y) \, \mathrm{d}y
$$

saying that $[b, A] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$.

For the converse implication define, as before

$$
f(s,y) = A(Y(s;y)) - \partial_y Y(s;y) A(y) {}^t\partial_y Y(s;y), \quad s \in \mathbb{R}, \quad y \in \mathbb{R}^m.
$$

For any $U \in C_c^1(\mathbb{R}^m)$ we have

$$
\begin{aligned}
F_U(s) &:= \int_{\mathbb{R}^m} f(s,y) : U(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} A(y) : U(Y(-s;y)) \, \mathrm{d}y - \int_{\mathbb{R}^m} \partial_y Y(s;y) A(y) {}^t\partial_y Y(s;y) : U(y) \, \mathrm{d}y
\end{aligned}
$$

and thus

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} F_U(s) &= - \int_{\mathbb{R}^m} A(y) : (\ (b \cdot \nabla_y) U\ )_{-s} \, \mathrm{d}y - \int_{\mathbb{R}^m} \partial_y (b(Y(s;y))) A(y) {}^t\partial_y Y(s;y) : U(y) \, \mathrm{d}y \\
&\quad - \int_{\mathbb{R}^m} \partial_y Y(s;y) A(y) {}^t\partial_y (b(Y(s;y))) : U(y) \, \mathrm{d}y \\
&= - \int_{\mathbb{R}^m} A(y) : (b \cdot \nabla_y) U_{-s} \, \mathrm{d}y - \int_{\mathbb{R}^m} \partial_y b(Y(s;y)) \partial_y Y(s;y) A(y) {}^t\partial_y Y(s;y) : U \, \mathrm{d}y \\
&\quad - \int_{\mathbb{R}^m} \partial_y Y(s;y) A(y) {}^t\partial_y Y(s;y) {}^t\partial_y b(Y(s;y)) : U(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} \{\partial_y b(Y(s;y)) f(s,y) + f(s,y) {}^t\partial_y b(Y(s;y))\} : U(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} f(s,y) : \{{}^t\partial_y b(Y(s;y)) U(y) + U(y) \partial_y b(Y(s;y))\} \, \mathrm{d}y.
\end{aligned}
$$

The previous equality still holds true for $U \in C_c(\mathbb{R}^m)$, and our conclusion follows as in the proof of Proposition 3.4, by Gronwall lemma. $\square$

Proof. (of Proposition 3.9)

1. $\implies$ 2. By Proposition 3.8 we deduce that $A(Y(s;y)) = \partial_y Y(s;y) A(y) {}^t\partial_y Y(s;y)$. Using the change of variable $y \to Y(s;y)$ one gets

$$\begin{aligned}\int_{\mathbb{R}^m} A(y)\nabla_y u \cdot \nabla_y v \, \mathrm{d}y &= \int_{\mathbb{R}^m} A(Y(s;y))(\nabla_y u)(Y(s;y)) \cdot (\nabla_y v)(Y(s;y)) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^m} A(y) {}^t\partial_y Y(s;y)(\nabla_y u)(Y(s;y)) \cdot {}^t\partial_y Y(s;y)(\nabla_y v)(Y(s;y)) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^m} A(y)\nabla_y u_s \cdot \nabla_y v_s \, \mathrm{d}y.\end{aligned}$$

2. $\implies$ 3. Taking the derivative with respect to s at $s = 0$ of the constant function $s \to \int_{\mathbb{R}^m} A(y)\nabla_y u_s \cdot \nabla_y v_s \, \mathrm{d}y$ yields

$$\int_{\mathbb{R}^m} A(y)\nabla_y (b \cdot \nabla_y u) \cdot \nabla_y v \, \mathrm{d}y + \int_{\mathbb{R}^m} A(y)\nabla_y u \cdot \nabla_y (b \cdot \nabla_y v) \, \mathrm{d}y = 0.$$

3. $\implies$ 2. For any $u, v \in C^2_c(\mathbb{R}^m)$ we can write, thanks to 3. applied with the functions u_s, v_s

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^m} A(y)\nabla_y u_s \cdot \nabla_y v_s \, \mathrm{d}y &= \int_{\mathbb{R}^m} A(y)\nabla_y(\ (b \cdot \nabla_y u)_s) \cdot \nabla_y v_s \, \mathrm{d}y \\ &+ \int_{\mathbb{R}^m} A(y)\nabla_y u_s \cdot \nabla_y(\ (b \cdot \nabla_y v)_s) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^m} A(y)\nabla_y (b \cdot \nabla_y u_s) \cdot \nabla_y v_s \, \mathrm{d}y \\ &+ \int_{\mathbb{R}^m} A(y)\nabla_y u_s \cdot \nabla_y (b \cdot \nabla_y v_s) \, \mathrm{d}y = 0.\end{aligned}$$

Therefore the function $s \to \int_{\mathbb{R}^m} A(y)\nabla_y u_s \cdot \nabla_y v_s \, \mathrm{d}y$ is constant on $\mathbb{R}$ and thus

$$\int_{\mathbb{R}^m} A(y)\nabla_y u_s \cdot \nabla_y v_s \, \mathrm{d}y = \int_{\mathbb{R}^m} A(y)\nabla_y u \cdot \nabla_y v \, \mathrm{d}y, \quad s \in \mathbb{R}.$$

Up to now, the symmetry of the matrix $A(y)$ did not play any role. We only need it for the implication 2. $\implies$ 1.

2. $\implies$ 1. We have

$$\begin{aligned}\int_{\mathbb{R}^m} A(y)\nabla_y u \cdot \nabla_y v \, \mathrm{d}y &= \int_{\mathbb{R}^m} A(y)\nabla_y u_s \cdot \nabla_y v_s \, \mathrm{d}y \\ &= \int_{\mathbb{R}^m} A(y) {}^t\partial_y Y(s;y)(\nabla_y u)_s \cdot {}^t\partial_y Y(s;y)(\nabla_y v)_s \, \mathrm{d}y \\ &= \int_{\mathbb{R}^m} \partial_y Y(s;y) A(y) {}^t\partial_y Y(s;y)(\nabla_y u)_s \cdot (\nabla_y v)_s \, \mathrm{d}y \\ &= \int_{\mathbb{R}^m} (\partial_y Y A\, {}^t\partial_y Y)_{-s} \nabla_y u \cdot \nabla_y v \, \mathrm{d}y\end{aligned}$$

where $(\partial_y Y A {}^t\partial_y Y)_{-s} = \partial_y Y(s; Y(-s;y)) A(Y(-s;y)) {}^t\partial_y Y(s; Y(-s;y))$. We deduce that

$$\int_{\mathbb{R}^m} (A(y) - (\partial_y Y A\, {}^t\partial_y Y)_{-s}) \nabla_y u \cdot \nabla_y v \, \mathrm{d}y = 0, \quad u, v \in C^1_c(\mathbb{R}^m).$$

Since $A(y) - (\partial_y Y A\ {}^t\partial_y Y)_{-s}$ is symmetric, it is easily seen, cf. Lemma A.1 below, that $A(y) - (\partial_y Y A\ {}^t\partial_y Y)_{-s} = 0$. Therefore we have $A(Y(s;y)) = \partial_y Y(s;y) A(y) {}^t\partial_y Y(s;y)$, $s \in \mathbb{R}, y \in \mathbb{R}^m$ and by Proposition 3.8 we deduce that $[b, A] = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. $\square$

Proof. (of Proposition 3.14)

1. Consider $A \in \mathrm{dom}(L), {}^tA = A$. It is easily seean that ${}^tA^{Q\pm} = A^{Q\pm}$ and

$$\begin{aligned}
|A^{Q+}|_Q^2 + |A^{Q-}|_Q^2 &= \int_{\mathbb{R}^m} (Q^{1/2} A Q^{1/2})^+ : (Q^{1/2} A Q^{1/2})^+ \,\mathrm{d}y \\
&+ \int_{\mathbb{R}^m} (Q^{1/2} A Q^{1/2})^- : (Q^{1/2} A Q^{1/2})^- \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} Q^{1/2} A Q^{1/2} : Q^{1/2} A Q^{1/2} \,\mathrm{d}y = |A|_Q^2 < +\infty
\end{aligned}$$

and therefore $A^{Q\pm} \in H_Q$. The positive/negative parts $A^{Q\pm}$ are orthogonal in H_Q

$$(A^{Q+}, A^{Q-})_Q = \int_{\mathbb{R}^m} (Q^{1/2} A Q^{1/2})^+ : (Q^{1/2} A Q^{1/2})^- \,\mathrm{d}y = 0.$$

We claim that $A^{Q\pm}$ satisfies (43). Indeed, thanks to (42) we can write, using the notation $X^{:2} = X : X$

$$\begin{aligned}
|G(s)A^{Q\pm} - A^{Q\pm}|_Q^2 &= \int_{\mathbb{R}^m} \{Q^{1/2} (\partial_y Y^{-1} (A^{Q\pm})_s {}^t\partial_y Y^{-1} - A^{Q\pm}) Q^{1/2}\}^{:2} \,\mathrm{d}y \qquad (64) \\
&= \int_{\mathbb{R}^m} \{{}^t\mathcal{O}(s;y) Q_s^{1/2} (A^{Q\pm})_s Q_s^{1/2} \mathcal{O}(s;y) - Q^{1/2} A^{Q\pm} Q^{1/2}\}^{:2} \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} \{{}^t\mathcal{O}(s;y) (Q_s^{1/2} A_s Q_s^{1/2})^{\pm} \mathcal{O}(s;y) - (Q^{1/2} A Q^{1/2})^{\pm}\}^{:2} \,\mathrm{d}y.
\end{aligned}$$

Similarly we obtain

$$|G(s)A - A|_Q^2 = \int_{\mathbb{R}^m} \{{}^t\mathcal{O}(s;y) Q_s^{1/2} A_s Q_s^{1/2} \mathcal{O}(s;y) - Q^{1/2} A Q^{1/2}\}^{:2} \,\mathrm{d}y. \qquad (65)$$

We are done if we prove that for any symmetric matrices U, V and any orthogonal matrix R we have the inequality

$$(\ {}^tRU^{\pm}R - V^{\pm}\ ) : (\ {}^tRU^{\pm}R - V^{\pm}\ ) \leq (\ {}^tRUR - V\ ) : (\ {}^tRUR - V\ ). \qquad (66)$$

For the sake of the presentation, we consider the case of positive parts U^+, V^+. The other one comes in a similar way. The above inequality reduces to

$$2\,{}^tRUR : V - 2\,{}^tRU^+R : V^+ \leq {}^tRU^-R : {}^tRU^-R + V^- : V^-$$

or equivalently, replacing U by $U^+ - U^-$ and V by $V^+ - V^-$, to

$$-2\,{}^tRU^+R : V^- - 2\,{}^tRU^-R : V^+ + 2\,{}^tRU^-R : V^- \leq {}^tRU^-R : {}^tRU^-R + V^- : V^-.$$

It is easily seen that the previous inequality holds true, since ${}^tRU^+R : V^- \geq 0$, ${}^tRU^-R : V^+ \geq 0$ and

$$2\,{}^tRU^-R : V^- \leq 2({}^tRU^-R : {}^tRU^-R)^{1/2}(V^- : V^-)^{1/2} \leq {}^tRU^-R : {}^tRU^-R + V^- : V^-.$$

Combining (64), (65) and (66) with

$$U = Q_s^{1/2} A_s Q_s^{1/2}, \;\; V = Q^{1/2} A Q^{1/2}, \;\; R = \mathcal{O}$$

yields

$$\sup_{s\neq 0} \frac{|G(s)A^{Q\pm} - A^{Q\pm}|_Q}{|s|} \leq \sup_{s\neq 0} \frac{|G(s)A - A|_Q}{|s|} \leq |L(A)|_Q$$

saying that $A^{Q\pm} \in \mathrm{dom}(L)$.

2. For any $A \in \mathrm{dom}(L)$, ${}^tA = A$ we can write

$$\begin{aligned}(A^{Q+}, A^{Q-})_Q &= \int_{\mathbb{R}^m} Q^{1/2} A^{Q+} Q^{1/2} : Q^{1/2} A^{Q-} Q^{1/2}\;\mathrm{d}y \\ &= \int_{\mathbb{R}^m} (Q^{1/2} A Q^{1/2})^+ : (Q^{1/2} A Q^{1/2})^-\;\mathrm{d}y = 0.\end{aligned}$$

Since $A^{Q\pm} \in \mathrm{dom}(L)$ we have

$$L(A^{Q\pm}) = \lim_{s\to 0} \frac{G(s/2)A^{Q\pm} - G(-s/2)A^{Q\pm}}{s}$$

and therefore, thanks to (42), we obtain

$$\begin{aligned}
&(L(A^{Q+}), L(A^{Q-}))_Q = \lim_{s\to 0} \left(\frac{G(\frac{s}{2})A^{Q+} - G(-\frac{s}{2})A^{Q+}}{s}, \frac{G(\frac{s}{2})A^{Q-} - G(-\frac{s}{2})A^{Q-}}{s} \right)_Q \\
&= \lim_{s\to 0} \int_{\mathbb{R}^m} \frac{Q^{1/2}(\;G(\frac{s}{2})A^{Q+} - G(-\frac{s}{2})A^{Q+}\;)Q^{1/2}}{s} : \frac{Q^{1/2}(\;G(\frac{s}{2})A^{Q-} - G(-\frac{s}{2})A^{Q-}\;)Q^{1/2}}{s}\;\mathrm{d}y \\
&= \lim_{s\to 0} \int_{\mathbb{R}^m} \frac{{}^t\mathcal{O}(\frac{s}{2};y)(Q_{\frac{s}{2}}^{1/2} A_{\frac{s}{2}} Q_{\frac{s}{2}}^{1/2})^+ \mathcal{O}(\frac{s}{2};y) - {}^t\mathcal{O}(-\frac{s}{2};y)(Q_{-\frac{s}{2}}^{1/2} A_{-\frac{s}{2}} Q_{-\frac{s}{2}}^{1/2})^+ \mathcal{O}(-\frac{s}{2};y)}{s} \\
&: \frac{{}^t\mathcal{O}(\frac{s}{2};y)(Q_{\frac{s}{2}}^{1/2} A_{\frac{s}{2}} Q_{\frac{s}{2}}^{1/2})^- \mathcal{O}(\frac{s}{2};y) - {}^t\mathcal{O}(-\frac{s}{2};y)(Q_{-\frac{s}{2}}^{1/2} A_{-\frac{s}{2}} Q_{-\frac{s}{2}}^{1/2})^- \mathcal{O}(-\frac{s}{2};y)}{s}\;\mathrm{d}y \\
&= -\lim_{s\to 0} \int_{\mathbb{R}^m} \frac{{}^t\mathcal{O}(\frac{s}{2};y)(Q_{\frac{s}{2}}^{1/2} A_{\frac{s}{2}} Q_{\frac{s}{2}}^{1/2})^+ \mathcal{O}(\frac{s}{2};y) : {}^t\mathcal{O}(-\frac{s}{2};y)(Q_{-\frac{s}{2}}^{1/2} A_{-\frac{s}{2}} Q_{-\frac{s}{2}}^{1/2})^- \mathcal{O}(-\frac{s}{2};y)}{s^2}\;\mathrm{d}y \\
&- \lim_{s\to 0} \int_{\mathbb{R}^m} \frac{{}^t\mathcal{O}(-\frac{s}{2};y)(Q_{-\frac{s}{2}}^{1/2} A_{-\frac{s}{2}} Q_{-\frac{s}{2}}^{1/2})^+ \mathcal{O}(-\frac{s}{2};y) : {}^t\mathcal{O}(\frac{s}{2};y)(Q_{\frac{s}{2}}^{1/2} A_{\frac{s}{2}} Q_{\frac{s}{2}}^{1/2})^- \mathcal{O}(\frac{s}{2};y)}{s^2}\;\mathrm{d}y \\
&\leq 0
\end{aligned}$$

since

$${}^t\mathcal{O}(\pm s/2;\cdot)(Q^{1/2}AQ^{1/2})^{\pm}_{\pm s/2}\mathcal{O}(\pm s/2;\cdot) \geq 0, \;\; {}^t\mathcal{O}(\mp s/2;\cdot)(Q^{1/2}AQ^{1/2})^{\pm}_{\mp s/2}\mathcal{O}(\mp s/2;\cdot) \geq 0.$$

$\square$

Proof. (of Proposition 5.3)

For any smooth functions $u, \varphi \in \ker(b \cdot \nabla_y)$ we have, cf. Lemma 5.1

$$\int_{\mathbb{R}^m} \langle \mathrm{div}_y(E\nabla_y(\mathrm{div}_y(E\nabla_y u)))\rangle \, \varphi \, \mathrm{d}y = \int_{\mathbb{R}^m} \mathrm{div}_y(E\nabla_y(\mathrm{div}_y(E\nabla_y u)))\varphi \, \mathrm{d}y$$

$$= \int_{\mathbb{R}^m} \mathrm{div}_y(E\nabla_y u) \, \mathrm{div}_y(E\nabla_y \varphi) \, \mathrm{d}y$$

$$= \int_{\mathbb{R}^m} \{\mathrm{div}_y(R\,{}^tE)\cdot({}^tR^{-1}\nabla_y u) + RE\,{}^tR : ({}^tR^{-1}\nabla_y \otimes {}^tR^{-1}\nabla_y)u\}$$

$$\times \{\mathrm{div}_y(R\,{}^tE)\cdot({}^tR^{-1}\nabla_y \varphi) + RE\,{}^tR : ({}^tR^{-1}\nabla_y \otimes {}^tR^{-1}\nabla_y)\varphi\} \, \mathrm{d}y$$

$$= \int_{\mathbb{R}^m} [\mathrm{div}_y(R\ {}^tE)\otimes \mathrm{div}_y(R\ {}^tE)] : [{}^tR^{-1}\nabla_y u \otimes {}^tR^{-1}\nabla_y \varphi] \, \mathrm{d}y$$

$$+ \int_{\mathbb{R}^m} [RE\ {}^tR \otimes \mathrm{div}_y(R\ {}^tE)] : [({}^tR^{-1}\nabla_y \otimes {}^tR^{-1}\nabla_y)u \otimes {}^tR^{-1}\nabla_y \varphi] \, \mathrm{d}y$$

$$+ \int_{\mathbb{R}^m} [\mathrm{div}_y(R\ {}^tE)\otimes RE\ {}^tR] : [({}^tR^{-1}\nabla_y u)\otimes ({}^tR^{-1}\nabla_y \otimes {}^tR^{-1}\nabla_y)\varphi] \, \mathrm{d}y$$

$$+ \int_{\mathbb{R}^m} [RE\ {}^tR \otimes RE\ {}^tR] : [({}^tR^{-1}\nabla_y \otimes {}^tR^{-1}\nabla_y)u \otimes ({}^tR^{-1}\nabla_y \otimes {}^tR^{-1}\nabla_y)\varphi] \, \mathrm{d}y$$

Recall that ${}^tR^{-1}\nabla_y$ leaves invariant $\ker(b\cdot\nabla_y)$ and therefore

$${}^tR^{-1}\nabla_y u \otimes {}^tR^{-1}\nabla_y \varphi \in \ker(b\cdot\nabla_y)$$

implying that

$$\int_{\mathbb{R}^m} [\mathrm{div}_y(R\ {}^tE)\otimes \mathrm{div}_y(R\ {}^tE)] : [{}^tR^{-1}\nabla_y u \otimes {}^tR^{-1}\nabla_y \varphi] \, \mathrm{d}y$$
$$= \int_{\mathbb{R}^m} \big\langle \mathrm{div}_y(R\ {}^tE)\otimes \mathrm{div}_y(R\ {}^tE)\big\rangle : [{}^tR^{-1}\nabla_y u \otimes {}^tR^{-1}\nabla_y \varphi] \, \mathrm{d}y.$$

Similar transformations apply to the other three integrals above, and finally one gets

$$\begin{aligned}\int_{\mathbb{R}^m} \langle \mathrm{div}_y(E\nabla_y(\mathrm{div}_y(E\nabla_y u)))\rangle \, \varphi \, \mathrm{d}y &= \int_{\mathbb{R}^m} X : [\nabla^R u \otimes \nabla^R \varphi] \, \mathrm{d}y \\ &+ \int_{\mathbb{R}^m} Y : [(\nabla^R \otimes \nabla^R)u \otimes \nabla^R\varphi] \, \mathrm{d}y \\ &+ \int_{\mathbb{R}^m} Z : [\nabla^R u \otimes (\nabla^R \otimes \nabla^R)\varphi] \, \mathrm{d}y \\ &+ \int_{\mathbb{R}^m} T : [(\nabla^R \otimes \nabla^R)u \otimes (\nabla^R \otimes \nabla^R)\varphi] \, \mathrm{d}y \\ &= I_1(u,\varphi) + I_2(u,\varphi) + I_3(u,\varphi) + I_4(u,\varphi)\end{aligned}$$

where $\nabla^R := {}^tR^{-1}\nabla_y$ and X, Y, Z, T are tensors of order two, three, three and four respectively

$$X_{ij} = \big\langle \mathrm{div}_y(R\ {}^tE)_i \, \mathrm{div}_y(R\ {}^tE)_j \big\rangle, \ \ i,j \in \{1,...,m\}$$

$$Y_{ijk} = \big\langle (RE\ {}^tR)_{ij} \, \mathrm{div}_y(R\ {}^tE)_k \big\rangle, \ \ Z_{ijk} = \big\langle \mathrm{div}_y(R\ {}^tE)_i \ \ (RE\ {}^tR)_{jk} \big\rangle, \ \ i,j,k \in \{1,...,m\}$$

$$T_{ijkl} = \left\langle (RE\ \ {}^tR)_{ij}\ \ (RE\ \ {}^tR)_{kl} \right\rangle, \ \ i,j,k,l \in \{1,...,m\}.$$

Integrating by parts one gets

$$I_1(u,\varphi) = \int_{\mathbb{R}^m} X\nabla^R u \cdot \nabla^R \varphi \, \mathrm{d}y = \int_{\mathbb{R}^m} R^{-1} X \nabla^R u \cdot \nabla_y \varphi \, \mathrm{d}y = \int_{\mathbb{R}^m} S_1(u)\varphi \, \mathrm{d}y$$

where $S_1(u) = -\mathrm{div}_y(R^{-1}X\nabla^R u)$. Notice that the differential operator

$$\xi \to \mathrm{div}_y(R^{-1}\xi) = \mathrm{div}_y({}^tR^{-1}) \cdot \xi + {}^t R^{-1} : \partial_y \xi$$

maps $(\ker(b\cdot\nabla_y))^m$ to $\ker(b\cdot\nabla_y)$, since the columns of R^{-1} contain fields in involution with b, and therefore S_1 leaves invariant $\ker(b\cdot\nabla_y)$, that is, for any $u \in \ker(b\cdot\nabla_y)$, $\xi = X\nabla^R u \in (\ker(b\cdot\nabla_y))^m$ and $S_1(u) = -\mathrm{div}_y(R^{-1}X\nabla^R u) = -\mathrm{div}_y(R^{-1}\xi) \in \ker(b\cdot\nabla_y)$. Similarly we obtain

$$I_2(u,\varphi) = \int_{\mathbb{R}^m} S_2(u)\varphi \, \mathrm{d}y, \ \ I_3(u,\varphi) = \int_{\mathbb{R}^m} S_3(u)\varphi \, \mathrm{d}y, \ \ I_4(u,\varphi) = \int_{\mathbb{R}^m} S_4(u)\varphi \, \mathrm{d}y$$

where S_2, S_3, S_4 are differential operators of order three, three and four respectively, which leave invariant $\ker(b\cdot\nabla_y)$. We deduce that

$$\int_{\mathbb{R}^m} \langle \mathrm{div}_y(E\nabla_y(\mathrm{div}_y(E\nabla_y u)))\rangle \, \varphi \, \mathrm{d}y = \int_{\mathbb{R}^m} S(u)\varphi \, \mathrm{d}y$$

for any $u, \varphi \in \ker(b\cdot\nabla_y)$, with $S = S_1 + S_2 + S_3 + S_4$, saying that

$$\langle \mathrm{div}_y(E\nabla_y(\mathrm{div}_y(E\nabla_y u)))\rangle - S(u) \perp \ker(b\cdot\nabla_y).$$

But we also know that

$$\langle \mathrm{div}_y(E\nabla_y(\mathrm{div}_y(E\nabla_y u)))\rangle - S(u) \in \ker(b\cdot\nabla_y)$$

and thus (62) holds true. $\square$

Lemma A.1 *Consider a field $A(y) \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ of symmetric matrices satisfying*

$$\int_{\mathbb{R}^m} A(y)\nabla_y u \cdot \nabla_y v \, \mathrm{d}y = 0, \ \ u, v \in C^1_c(\mathbb{R}^m). \tag{67}$$

Therefore $A(y) = 0$ a.a. $y \in \mathbb{R}^m$.

Proof. Applying (67) with $v_j = y_j v$, $v \in C^1_c(\mathbb{R}^m)$, $u_i = y_i\varphi(y)$ where $\varphi \in C^1_c(\mathbb{R}^m)$ and $\varphi = 1$ on the support of v, yields

$$\int_{\mathbb{R}^m} A(y)e_i \cdot (y_j \nabla_y v + v e_j) \, \mathrm{d}y = 0. \tag{68}$$

Applying (67) with v and $u_{ij} = y_i y_j \varphi(y)$ one gets

$$\int_{\mathbb{R}^m} A(y)(y_j e_i + y_i e_j) \cdot \nabla_y v \, \mathrm{d}y = 0. \tag{69}$$

Combining (68), (69) we obtin for any $i, j \in \{1, ..., m\}$

$$2 \int_{\mathbb{R}^m} (A(y)e_i \cdot e_j) \, v(y) \, \mathrm{d}y = \int_{\mathbb{R}^m} (A(y)e_i \cdot e_j + A(y)e_j \cdot e_i) v(y) \, \mathrm{d}y = 0$$

saying that $A(y) = 0$, a.a. $y \in \mathbb{R}^m$. $\square$

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